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STABILITY OF A RAYLEIGH-BENARD POISEUILLE FLOW CONSIDERING A CARREAU FLUID

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Abstract. *The present work studies through a linear stability analysis, the transition of the stable to the convective unstable state of the Rayleigh-Benard-Poiseuille problem considering a non-Newtonian fluid of the Carreau kind. This analysis consists in perturbing the problem's stationary solution and observes the spatial and temporal perturbation behavior. The resulting problem consists in a system of differential equations which are treated as an eigenvalue problem. This problem is solved with a shooting method using the Mathematica software. The Carreau model was chosen because it is used to describe several fluids with industrial applications on the aliment, pharm, oil and other fields.*

Keywords: *Non-Newtonian fluid mix convection; convective instability; Carreau Fluid*

1. INTRODUCTION

The classic Rayleigh-Benard problem consists on the natural convection of a stead fluid layer been heated from below. This convective process takes the name of Rayleigh-Benard convection. Since the first studies developed in parallel for Lord Rayleigh and Henri Benard, this kind of problem has been expended and widely studied. Most of this works are developed considering Newtonian fluids, however it is known that a considerable quantity of fluids presented in nature and in industrials applications do not follow this behavior. Therefore, it is interesting the development of scientific works which analyses the behavior of those kind of fluids.

Vest and Arpaci (1969) and Sokolov and Tanner (1972) studied the convective instability onset using the linear stability analyses. Tsuei and Tien (1973) used aqueous solutions of carboxymethylcellulose and carbopol to experimentally analyze the natural convection of non-Newtonian fluids.

An important variation of the classic Rayleigh-Benard problem is the Rayleigh-Benard-Poiseuille (RBP) problem, in which the fluid layer moves due to a horizontal pressure gradient. The fluid layer velocity profile has a parabolic behavior for Newtonian fluids, however it will be shown in the present work that it is a quite different for Carreau fluids. Gage and Reid (1968) presents the first linear stability analyses study for the RBP problem which analyses the onset of the convective instabilities. Moreover, Carriere and Monkewitz (1999) studied the transition of the convective instability to the absolute one. Nicolas (2002) presents a bibliographic review about the RBP considering Newtonian fluids.

Even if there is a wide contribution of works involving RBP problems with Newtonian fluids, the development of works considering non-Newtonian ones is new. Métivier and Nouar (2008) is the first work considering a viscoplastic fluid in a RBP problem through a modal and a non-modal analysis is done to find the transition to the convective instability. Hirata et al. (2015) is the first work where the onset of the convective and the onset of the absolute instabilities of viscoelastic fluids are presented.

Only few works involving the Rayleigh-Benard convection and its variations considering the fluids of Carreau are present on the literature. Some examples are the Bouteraa et al. (2015), which performs a linear and weakly non-linear stability analysis of the Rayleigh-Benard convection of a Carreau fluid. Jenny et al. (2015) presents a non-linear analysis of the sub critic Rayleigh-Benard convection roller. Darbouli et al. (2016) experimentally analysis the natural convection of a Carreau fluid through the magnetic resonance imaging technique.

It is not known the existence of any work on the literature presenting an analysis of the RBP problem considering a Carreau Fluid. Therefore, the present work has the objective of study the transition to the convective instability of the

Raleigh-Benardd-Poiseuille mix convection considering a non-Newtonian fluid modeled with the Carreau equation. It was chosen because it is applied in many fluids with industrial applications.

2. MATHEMATICAL FORMULATION

2.1 Dimentionless Quantities

The dimensionless quantities used in this work are presented below:

$$(x, y, z) = \frac{(x^*, y^*, z^*)}{h}, \qquad \mathbf{u} = \frac{\mathbf{u}^*}{K/h}, \qquad t = \frac{t^*}{h^2/K}, \qquad T = \frac{T^*}{\Delta T^*}, \qquad p = \frac{p^*}{\mu_0^* K/h^2} \tag{1}$$

$$Ra = \frac{\rho_0 g \beta \Delta T^* h^3}{\mu_0^* K}, \qquad Re = \frac{\rho_0 u_0^* h}{\mu_0^*}, \qquad Pr = \frac{\mu_0^*}{\rho_0 K}, \tag{2}$$

here starred quantities are dimensional, h is the distance between plates, K is the fluid's thermal diffusivity, $\mathbf{u}$ is the velocity vector, T^* is the temperature, ΔT^* is the difference between the temperature on the bottom and the top plate, p is the hydrostatic pressure, μ_0^* is the dynamic viscosity of the Carreau fluid at no shear rate, g is the gravitational acceleration, β is the fluid's expansion coefficient, ρ_0 is the fluid's specific mass at the upper surface temperature, u_0^* is the maximum velocity on the base flow velocity profile, Ra, Re and Pr are the Rayleigh, Reynolds and Prandtl numbers.

2.2 Governing Equations

The governing equations are written in a dimensionless form according to the quantities:

$$\nabla \cdot \mathbf{u} = 0, \tag{3}$$

$$\frac{\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}}{Pr} = -\nabla p + \nabla \cdot \boldsymbol{\tau} + Ra\theta \mathbf{e_z}, \tag{4}$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = \nabla^2 \theta, \tag{5}$$

$$\begin{aligned} z &= \ \ 1/2, \qquad \mathbf{u} = (0,0,0), \qquad \theta = 0, \\ z &= -1/2, \qquad \mathbf{u} = (0,0,0), \qquad \theta = 1, \end{aligned} \tag{6}$$

here Eq. (3) is the continuity equation, Eq. (4) is the momentum equation, where the Oberbeck-Boussinesq hypothesis is assumed and $\boldsymbol{\tau}$ is the stress tensor, Eq. (5) is the energy equation and Eq. (6) are the boundary conditions for the velocity and temperature at the upper and bottom plates, where θ is defined as $\theta = (T^* - T_U^*)/\Delta T^*$, with T^* as the local temperature and T_U^* as the upper surface temperature. The stress tensor is defined as:

$$\boldsymbol{\tau} = 2\mu \mathbf{D}, \tag{7}$$

where $\mathbf{D}$ is the strain rate tensor and is defined as the symmetric part of the velocity gradient tensor:

$$\mathbf{D} = \frac{\nabla \mathbf{u} + \nabla \mathbf{u}^T}{2}, \tag{8}$$

and μ is the dimensionless viscosity defined for the Carreau model, which is described in the next section.

3. REOLOGY

The Carreau model describes the dimensional dynamic viscosity μ^* as a function of the shear rate $\dot{\gamma}^*$ as follow:

$$\mu^*(\dot{\gamma}^*) = \mu_\infty^* + (\mu_0^* - \mu_\infty^*)[1 + (\lambda^* \dot{\gamma}^*)^2]^{\frac{n-1}{2}}. \tag{9}$$

In this model μ_∞^* is the viscosity under an infinite shear rate, n and λ^* are non-Newtonian parameters defined for the fluids properties. If λ^* is equal to zero or n is equal to the unity, the fluid has a Newtonian behavior and $\mu^*(\dot{\gamma}) = \mu_0^*$. The higher λ^* and smaller n, the more the viscosity will decrease as the shear rate increases.

The shear rate is function of the strain rate tensor and can be written as $\dot{\gamma}^* = \sqrt{2\mathrm{Tr}(\mathbf{D}^* \cdot \mathbf{D}^*)}$, therefore, its unit is s^{-1} and the unit of λ^* has to be s. The dimensionless form of the Carreau equation can be found by dividing Eq. (9) by μ_0^*:

$$\mu(\dot{\gamma}) = R + (1-R)[1+(\lambda\dot{\gamma})^2]^{\frac{n-1}{2}}, \tag{10}$$

where R is the relation μ_∞^*/μ_0^*, $\dot{\gamma}$ and λ are the dimensionless forms of $\dot{\gamma}^*$ and λ^*.

4. LINEAR STABILITY ANALISIS

4.1 Linearization

In order to evaluate the stability of the problem, small perturbations are imposed to the base state:

$$\begin{aligned} u(x,y,z,t) &= u_b(x,y,z) + \epsilon u_p(x,y,z,t), \\ v(x,y,z,t) &= v_b(x,y,z) + \epsilon v_p(x,y,z,t), \\ w(x,y,z,t) &= w_b(x,y,z) + \epsilon w_p(x,y,z,t), \\ p(x,y,z,t) &= p_b(x,y,z) + \epsilon p_p(x,y,z,t), \\ \theta(x,y,z,t) &= \theta_b(x,y,z) + \epsilon \theta_p(x,y,z,t), \end{aligned} \tag{11}$$

where ϵ is a small perturbation parameter.

4.2 Basic Solution

As the stead state of the problem formulation consists in a fully developed flow in the x direction, where the fluid is between stationary parallel plates. The flow is generated due to a pressure gradient in this direction. Moreover, the existence of a temperature gradient in the z direction, as the gravitational field is considered. Therefore, the stationary solution for the velocity and temperature field can be written as:

$$\begin{aligned} \mathbf{u}_b(z) &= (u_b(z), 0,0), \\ \theta_b(z) &= 1/2 - z. \end{aligned} \tag{12}$$

By taken the first component of Eq. (4) and making the material derivative equal to zero the following differential equation, which gives the base velocity profile, can be found:

$$\frac{d\big(u_b'(z)\mu_b(z)\big)}{dz} = \frac{\partial p}{\partial x}, \tag{13}$$

where:

$$\mu_b(z) = R + (1-R)(1+(\lambda\dot{\gamma}_b)^2)^{\frac{n-1}{2}}, \qquad \dot{\gamma}_b = u_b'(z), \qquad \frac{\partial p}{\partial x} = constant. \tag{14}$$

Integrating both sides of Eq. (13) in z and observing that due to the problem's symmetricity $u_b'(0) = 0$, Eq. (13) can be simplified as:

$$u_b'(z)\left(R + (1-R)(1+\lambda^2 u_b'(z)^2)^{\frac{n-1}{2}}\right) = \frac{\partial p}{\partial x} z. \tag{15}$$

By analyzing Eq. (15), it is clear that the basic state velocity profile for a Poiseuille flow considering a Carreau fluid is very dependent on the non-Newtonian parameters. It happens because the shear rate in this profile is affected for the viscosity which is affected for the shear rate. The solution of this equation is found using an interactive method which uses a guess on the velocity profile inside the viscosity function and a numerical method called shooting method which knowing the value of the velocity in the middle of the profile, finds the value of pressure gradient that makes the profile have the value of zero at the walls, $z = \pm 1/2$. Then the new profile is applied into the viscosity function and the procedure is repeated. The first guess to the velocity profile is the parabolic solution of the Newtonian case. The problem is considered converged when the integral of the difference between the previous and actual profile is smaller than some criteria. It is important to note that the dimensionless value to the velocity in the middle of the profile is the maximum velocity of the profile, which means:

$$u_b(0) = \frac{u_0^* h}{K} = RePr. \tag{16}$$

Figure 1, 2 and 3 shows the result obtained for the above approach for the profile of velocity with different values of λ, n and $RePr$.

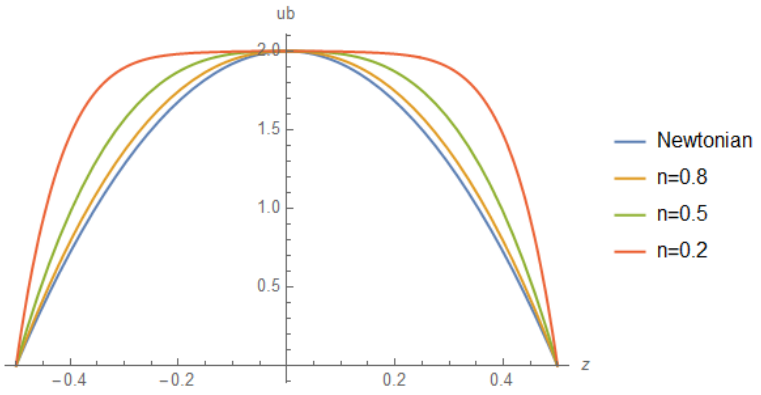


Figure 1. Velocity profile for $\lambda = 10$, $RePr = 2$ and the n values showed on the legend.

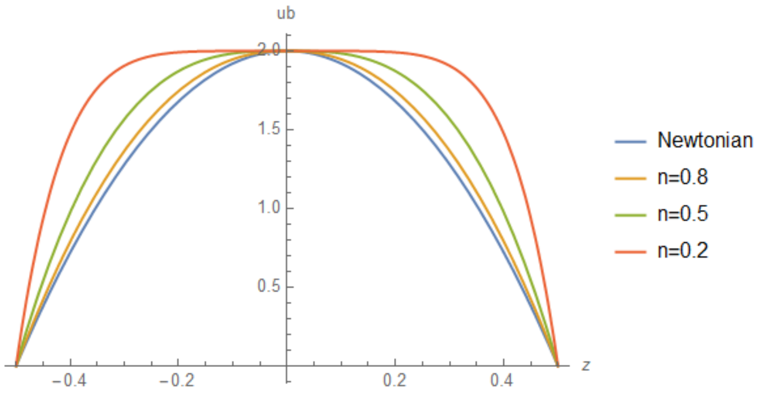


Figure 2. Velocity profile for $\lambda = 100$, $RePr = 2$ and the n values showed on the legend.

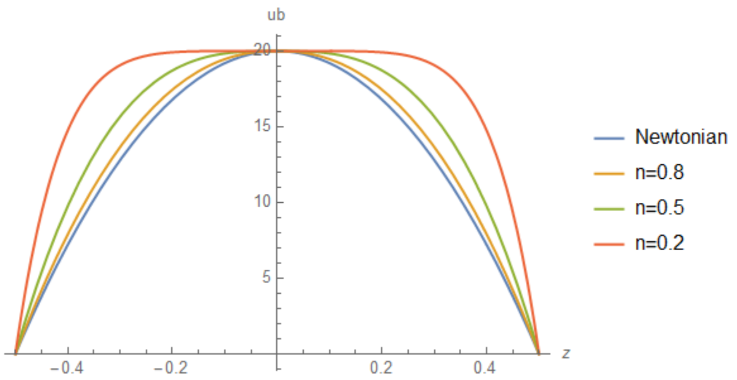


Figure 3. Velocity profile for $\lambda = 100$, $RePr = 20$ and the n values showed on the legend.

It is observed that, in the range of the evaluated properties, the velocity profile is more dependent on the changes in the n value than on the changes of the other parameters values. Moreover, it is noted that the decrease of the viscosity in the regions closer to the walls increases the velocity in this region making the profile been more squared.

Table 1 shows the number of iterations used to obtain the figures above with the convergence criterion of $RePr.10^{-8}$, where it is also found that based on the chosen parameters, the interaction number is highly dependent on n and not very dependent on the other parameters.

Table 1. Number of interactions needed to find the velocity profile considering different parameters.

n	$RePr = 2$	$RePr = 2$	$RePr = 20$
	$\lambda = 10$	$\lambda = 100$	$\lambda = 100$
0.2	11	10	10
0.5	23	23	23
0.8	64	66	66

4.3 The Linearized Perturbation Equation

By applying the perturbed solution in the governing equations and by taking the linear part of it, which means the terms with $O(\epsilon)$, the following system can be found for the perturbation equation:

$$\nabla \cdot \mathbf{u}_p = 0, \tag{17}$$

$$\frac{\frac{\partial \mathbf{u}_p}{\partial t} + \mathbf{u}_b \cdot \nabla \mathbf{u}_p + \mathbf{u}_p \cdot \nabla \mathbf{u}_b}{Pr} = -\nabla p_p + \nabla \cdot 2\mu_b(z)\mathbf{D}_p + Ra\theta_p \mathbf{e_z}, \tag{18}$$

$$\frac{\partial \theta_p}{\partial t} + \mathbf{u}_b \cdot \nabla \theta_P = \nabla^2 \theta_P, \tag{19}$$

$$z = \pm 1/2 \rightarrow \quad \mathbf{u}_p = (0,0,0), \qquad \theta_P = 0. \tag{20}$$

Therefore, taking the double curl of the momentum equation and using the continuity on the third component of the obtained equation, the pressure p and the velocity components u_p and v_p are taken out of the equation. Then, with the energy equation, we obtain a set of two coupled equations involving w_p, the vertical component of the perturbed velocity, and θ_p, the temperature perturbation:

$$\frac{1}{Pr}\left(\frac{\partial \nabla^2 w_p}{\partial t} + u_b \frac{\partial \nabla^2 w_p}{\partial x} - u_b'' \frac{\partial w_p}{\partial x}\right) = \mu_b''\left(\frac{\partial^2 w_p}{\partial z^2} - \nabla_\perp^2 w_p\right) + 2\mu_b' \frac{\partial \nabla^2 w_p}{\partial z} + \mu_b \nabla^4 w_p + Ra \nabla_\perp^2 \theta_p \tag{21}$$

$$\frac{\partial \theta_p}{\partial t} + u_b \frac{\partial \theta_p}{\partial x} = \nabla^2 \theta_p + w_p. \tag{22}$$

where $\nabla_\perp^2$ is the horizontal Laplacian (in the x, y plane). The fourth-order equation for w_p requires an additional boundary condition on its vertical derivatives which is deduced from the continuity equation leading to the conditions below:

$$z = \pm 1/2 \rightarrow \; w_p = \frac{\partial w_p}{\partial z} = \theta_p = 0. \tag{23}$$

4.4 Normal Modes

By rewriting the w_p and θ_p as:

$$w_p(x, y, z, t) = w_n(z) e^{i(\alpha x + \beta y - \omega t)}, \tag{24}$$

$$\theta_p(x, y, z, t) = \theta_n(z) e^{i(\alpha x + \beta y - \omega t)}, \tag{25}$$

the perturbation equation can be written for normal modes. Here the real part of α and β are the dimensionless wave numbers related to the x and y directions respectively and the real part of ω is the frequency. The imaginary part of those numbers are the spatial growth rate on the x and y direction and the temporal growth respectively. As we are only interested on the convective instability transition, its imaginary parts will be considered zero. If we focus only on the transversal modes, $\beta = 0$ and α can be treated as κ. Therefore, by applying Eq. (24) and Eq. (25) on Eq. (21) and Eq. (22) the disperse equations system is obtained:

$$\kappa^2[Ra\theta_n + 2(w_n'\mu_b)' - (\kappa^2\mu_b + \mu_b'')w_n] - (\mu_b w_n'')'' = i\frac{\kappa(u_b'' w_n + \kappa^2 u_b w_n - u_b w_n'') - \omega(\kappa^2 w_n - w_n'')}{Pr}, \tag{26}$$

$$w_n - \kappa^2 \theta_n + \theta_n'' = i(\kappa u_b - \omega), \tag{27}$$

where the boundary conditions can be found by applying Eqs. (24) and (25) to Eq. (23):

$$z = \pm 1/2 \rightarrow \ w_n = w_n' = \theta_n = 0. \tag{28}$$

5. SOLUTION OF THE EIGENVALUE PROBEM

The solution of the eigenvalue problem is here reached by using the numerical method called shooting method. This method consists in reducing a boundary value problem in an initial value problem and then solving it using a marching method for solving differential equations with a given initial condition. To transform the boundary condition problem in an initial value problem, the boundary conditions relative to $z = 1/2$ must be disregarded and new initial conditions at $z = -1/2$ must be created. In this case the new initial conditions are given by:

$$w_n = w_n' = 0, \qquad w_n'' = C_1, \qquad w_n''' = C_2 \tag{29}$$
$$\theta_n = 0, \qquad \theta_n' = C_3, \tag{30}$$

here the additional imaginary constants C_1, C_2 and C_3 must be chosen in such way to guarantee that the solutions w_n, w_n' and θ_n at $z = 1/2$. In the present work, the entire problem is re-scaled by C_3, what is equivalent to make $C_3 = 1$. Two unknown real parameters of the disperse equations can now be determined with C_1 and C_2, in the present work the chosen parameters are Ra and ω, while Re, Pr, κ are defined. Two numerical procedures provided by the Wolfarm Mathematica system are employed in this method. The first one is the built-in function *NDSolve* to march the ODEs from $z = -1/2$ to $z = 1/2$. Such procedure is programed into a function which gives a real vector with six elements $(\mathrm{Re}[w_n], \mathrm{Im}[w_n], \mathrm{Re}[w_n'], \mathrm{Im}[w_n'], \mathrm{Re}[\theta_n], \mathrm{Im}[\theta_n])$ evaluated at $z = 1/2$. The second one is the built-in function *FindRoot*, which finds the values of $(Ra, \omega, \mathrm{Re}[C_1], \mathrm{Im}[C_1], \mathrm{Re}[C_2], \mathrm{Im}[C_2])$ which gives the root of the function described in the previous sentence. It means, gives the values to the vector $(\mathrm{Re}[w_n], \mathrm{Im}[w_n], \mathrm{Re}[w_n'], \mathrm{Im}[w_n'], \mathrm{Re}[\theta_n], \mathrm{Im}[\theta_n])$ be close to zero according to the tolerance prescribed.

6. RESULTS AND DISCUSSION

We present some results for the onset of convective instability in this section. This paper focused on the analysis of the transversal modes of instabilities. From figure 4 to 6 the marginal stability curves - Rayleigh number Ra versus wave number κ and frequency ω versus wave number κ - are shown for different combinations of the Carreau model parameters with Prandtl number equal to the unit. It is possible to see that for the Newtonian case, the Reynolds almost does not change the marginal stabilities curves and that as de fluid becomes more non-Newtonian, when n decreases, the fluids flow is destabilized as the Reynolds increases.

Figure 4 presents the marginal stabilities curves for different values of Reynolds numbers for the Newtonian case with the Prandtl number equal to the unit. Figure 5 presents the marginal stability curves for a non-Newtonian case where $\lambda = 10, n = 0.8, R = 0$ and $Pr = 1$ while Figure 6 presents the marginal stability curves for a non-Newtonian case where $\lambda = 10, n = 0.5, R = 0$ and $Pr = 1$ for different Reynolds number. By changing n is possible to see the different behavior of the marginal stability curves.

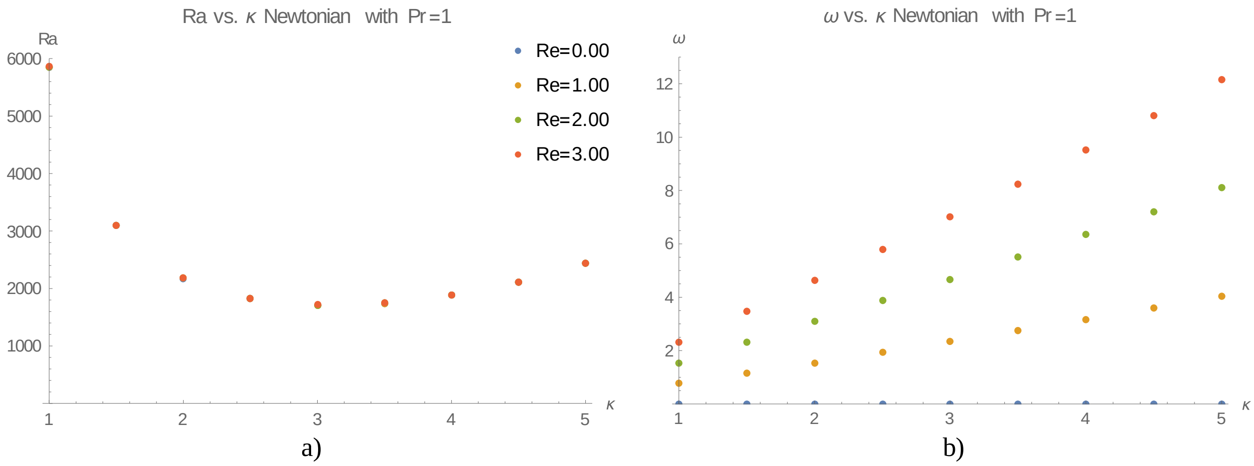
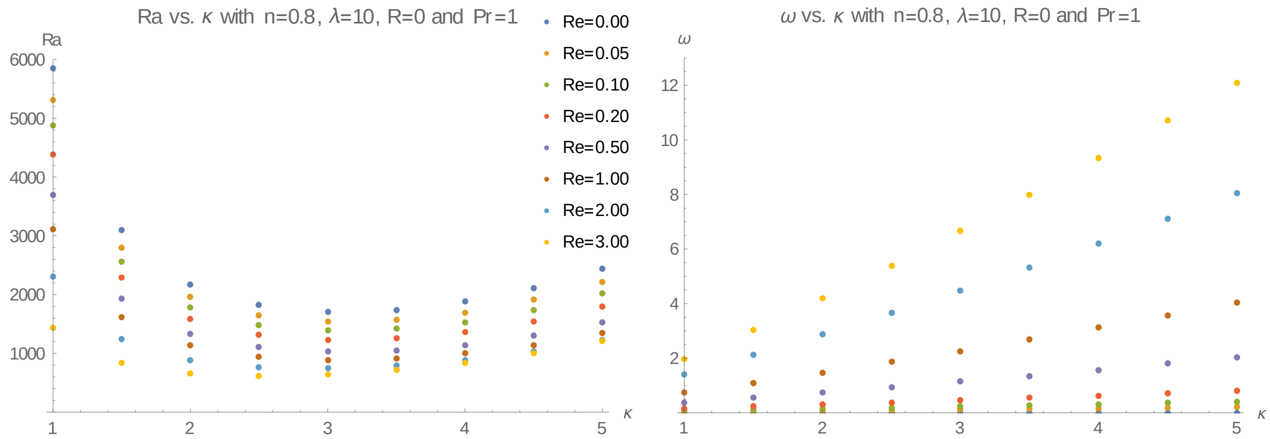
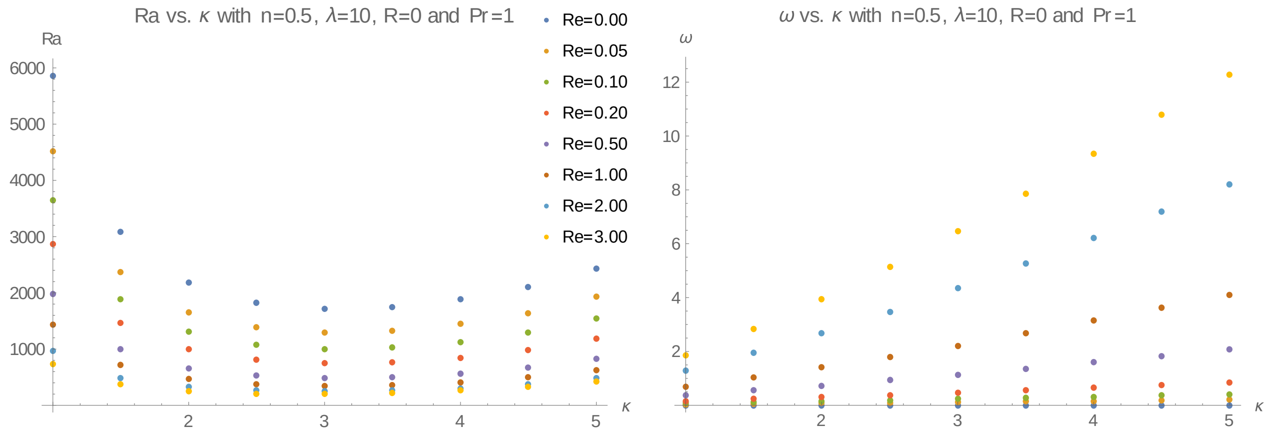


Figure 4. Marginal Stabilities Curves for the Newtonian case, with $Pr = 1$ and different Reynolds a) Ra and b) ω.



a) b)

Figure 5. Marginal Stabilities Curves for $n = 0.8$, $\lambda = 10$, $R = 0$ and $Pr = 1$ for different values of Reynolds a) Ra and b) ω.



a) b)

Figure 6. Marginal Stabilities Curves for $n = 0.5$, $\lambda = 10$, $R = 0$ and $Pr = 1$ for different values of Reynolds a) Ra and b) ω.

The figures above show that for $Re = 0$ the marginal stability curves are the same for the Newtonian and non-Newtonian cases, as was explained in previous sections. More over it is noted that for low values of the Prandtl number the transitional Rayleigh number decreases for each Reynolds and consequently the critical Rayleigh number. It is the opposite behavior than the one observed on Newtonian fluid case. In addition, it is found the transitional frequency increases with Reynolds, Prandtl and wave number as it happens for the Newtonian case.

7. CONCLUSIONS

This paper presents the onset of convective instabilities on the Rayleigh-Benard-Poisseullie problem with a non-Newtonian fluid of the Carreau kind. The problem consists in a fluid flow, imposed for a pressure gradient, between parallel plates, heated from below. The analyses present the transition of the situation where the buoyancy forces and the viscous forces are more significant, and the Raleigh-Benard convection rolls can be created. It was found that differently from the Newtonian case, the non-Newtonian case is destabilized as the Reynolds number increases from zero, the transitional Rayleigh decreases for each wave number as well as the critical Rayleigh number.

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