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A matrix solver approach for simulation of circular inhomogeneities by the Analytic Element Method

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Abstract. *This work deals with the study of two-dimensional groundwater flow of circular inhomogeneities by the Analytic Element Method. Each circular inhomogeneity has the influence that can be expanded in a series which exactly satisfies the Laplace's equation. In series expansion, the unknown coefficients can be obtained from the discharge potentials of all other elements that are related with the coefficients of the expansion. For all inhomogeneities, locations, sizes and conductivities may be selected arbitrarily. A matrix approach obtained by imposing the intern boundary conditions for the Analytic Element Method also discusses in the prescribed work.*

Keywords: *Circular inhomogeneities, analytic element method, groundwater flow, matrix method.*

1. Introduction

Problems associated with groundwater flow have been an area of active research for the past few decades. Researchers working in these areas strived for years to develop different numerical methods using the governing equations of the physical phenomena (Hussain, 2017; Marin, 2011; Strack, 1989; Haitjema, 1995). Otto Strack came up in 1989 with Analytic Element Method (AEM) as a method to simulate problems inherent to groundwater flow (Strack, 1989, 1999; Janković, 1997). This method is based on superposition principle, putting to use such solutions of the equations, that govern steady state flow, and satisfy boundary condition in the focus area in general. The equations governing the steady state flow, result in solutions termed as analytical elements, each presented by these equations (Janković and Barnes, 1999a; Marin, 2011; Strack, 1989, 1999). These solutions will be accurate, but will satisfy the boundary conditions rather approximately. The linear nature of the governing equation with respect to the discharge potential of each analytic element makes the AEM satisfy Laplace's equation (Janković and Barnes, 1999a; Marin, 2011; Strack, 1999, 1989). This method does not require the specific domain to be discretized giving rise to a grid. It only requires the boundary conditions at infinity and the internal conditions at certain points (Marin, 2011; Strack, 1999, 1989).

The superposition principle is used to write the complex potential in AEM, to find the problem a final solution. The superposition principle sums up for each physical feature the multiple complex potential components, within the model. The complex potential, $\Omega(z)$, at a point $z = x + iy$ in the complex plane can be expressed as:

$$\Omega(z) = \Phi(z) + i\Psi(z), \tag{1}$$

where $\Phi(z)$, the discharge potential, proportional to the hydraulic head, is the real part of the complex potential and $\Psi(z)$ is the associated stream function.

A circular inhomogeneity j with radius R_j and center c_j, has complex potential, as presented in different works (Barnes and Janković, 1999; Janković and Barnes, 1999a; Janković, 1997; Marin, 2011; Hussain, 2017):

$$\Omega_j(Z) = \begin{cases} a_0 + \sum\limits_{n=1}^{\infty} a_n Z_j^n & |Z| < 1 \\ -\sum\limits_{n=1}^{\infty} \overline{a}_n Z_j^{-n} & |Z| \geq 1 \end{cases}, \tag{2}$$

for which

$$Z = \frac{z - c_j}{R_j}, \tag{3}$$

where a_0, the coefficient, is real in nature while all the a_n are complex, z represents any position in the global coordinate system and $Z = X + iY$ describe a local coordinate system which represents similar position specified in local coordinates.

According to (Barnes and Janković, 1999; Marin, 2011), it is assumed that the circular inhomogeneity does not meet with other elements and the stream function $\Psi_j(Z)$ given by Eq. ((2) is continuous across the boundary due to $\Im(a_n e^{in\theta}) = -\Im(\overline{a}_n e^{-in\theta})$. The complex potential of all the elements present in domain can be represented as:

$$\Omega(Z) = \Omega_j(Z) + \Omega_{\neq j}(Z), \tag{4}$$

where $\Omega_{\neq j}(Z)$ is the combined complex potential of all circular inhomogeneities except j. The continuity of $\Omega_{\neq j}(Z)$ and Eq. (2) ensure the continuity of the stream function across the boundary of the circular inhomogeneity. Therefore, it is sufficient for the continuity of flow everywhere in the given domain.

2. Preliminaries

This work deals with the problem of an aquifer composed of circular regions whose hydraulic conductivity is different from the surrounding medium, shown in Figure 1.

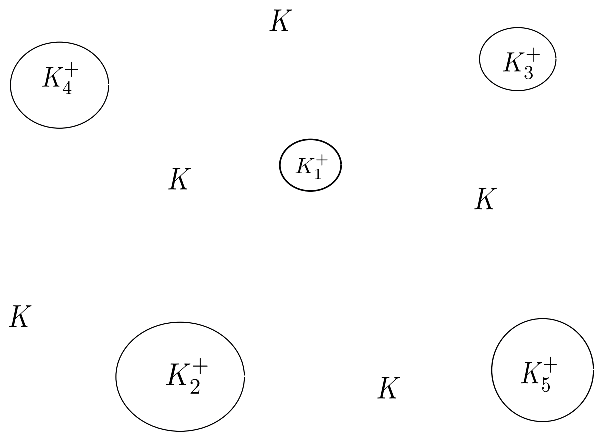

Figure 1. Aquifer with circular inhomogeneities of distinct hydraulic conductivities.

Circular inhomogeneities are circular regions with different hydraulic conductivities. Let K^+ be the hydraulic conductivity inside and K^- be the hydraulic conductivity outside the circle j, which shows the permeability of the material (Janković and Barnes, 1999a; Strack, 1999). To satisfy the boundary conditions, the head must be continuous at the edge of an inhomogeneity (Barnes and Janković, 1999), that is:

$$\phi^+(\theta) = \phi^-(\theta). \tag{5}$$

The only component that jumps across the boundary of the circular inhomogeneity is the component of the discharge potential itself, contributed by all circular inhomogeneities (Barnes and Janković, 1999; Janković and Barnes, 1999a; Hussain, 2017). Hence, the continuity condition of the head at the edge of an inhomogeneity can be written as:

$$\frac{\Phi^+}{K^+} = \frac{\Phi^-}{K^-}, \tag{6}$$

where the $+$ sign is the internal part and the $-$ sign is the external part of the inhomogeneity, whereas, the inside hydraulic conductivity is represented by K^+ and K^- is the hydraulic conductivity outside the circle j. Due to the continuity of the head, the boundary can be parametrized by the angle θ, i.e., $0 \leq \theta \leq 2\pi$, that leads to the relation:

$$\frac{\Phi^+(\theta)}{K^+} = \frac{\Phi^-(\theta)}{K^-}, \tag{7}$$

as presented in (Barnes and Janković, 1999; Marin, 2011). By combining the discharge potentials of all the components of circular inhomogeneities, we get:

$$\frac{\Phi^+_{\neq j}(\theta) + \Phi^+_j(\theta)}{K^+} = \frac{\Phi^-_{\neq j}(\theta) + \Phi^-_j(\theta)}{K^-}, \tag{8}$$

or

$$K^-\Phi^+_j(\theta) - K^+\Phi^-_j(\theta) = (K^+ - K^-)\Phi^-_{\neq j}(\theta) = (K^+ - K^-)\Phi_{\neq j}(\theta), \tag{9}$$

where $\Phi^+_j(\theta)$ is the discharge potential inside the circle j, $\Phi^-_j(\theta)$ is the discharge potential outside the circle j and $\Phi^+_{\neq j}(\theta)$ is the discharge potentail inside all circular inhomogeneities, whereas $\Phi^-_{\neq j}(\theta)$ is the discharge potentail outside all circular inhomogeneities except j.

Since, the discharge potential generated by other elements is continuous over the edge of circular inhomogeneity, its boundary conditions may be written as:

$$K^-\Phi_j^+(\theta) - K^+\Phi_j^-(\theta) = (K^+ - K^-)\Phi_{\neq j}(\theta), \tag{10}$$

which is used to determine expressions for calculating the unknown coefficients.

2.1 Determining the coefficients

To calculate the unknown coefficients of potential expansion for an analytical element, the boundary condition Eq. (5) for a circular inhomogeneity may be used. It may be calculated using Eq. (2) by the use of Fourier orthogonal relationship. Substituting the real part of Eq. (2) in Eq. (10), multiplying both sides by $e^{-im\theta}$, and integrating from 0 to 2π (Barnes and Janković, 1999; Marin, 2011), we get:

$$a_0 = \left(\frac{K^+ - K^-}{K^-}\right)\frac{1}{2\pi}\int_0^{2\pi}\Phi_{\neq j}(\theta)d\theta \tag{11}$$

and

$$a_n = \left(\frac{K^+ - K^-}{K^+ + K^-}\right)\frac{1}{\pi}\int_0^{2\pi}\Phi_{\neq j}(\theta)e^{-in\theta}d\theta, \tag{12}$$

where a_0 is real and all a_n are complex.

Using Eq. (2) into Eq. (12) we get an infinite linear system for the unknowns $a_0, a_1, \ldots$. Obviously, in practice the infinite series must be truncated and only a finite number of coefficients have to be found.

2.2 Exact solution

The exact solution of the problem of a circular inhomogeneity having radius R, centred at origin in a uniform flow field can be determined by (Strack, 2014; Marin, 2011; Hussain, 2017):

$$\Omega^+(Z) = -Q_{x_0} z\frac{2K^+}{K^+ + K^-} + \frac{K^+}{K^-}\Phi_0 \qquad \text{for} \qquad z\overline{z} < R^2 \tag{13}$$

and

$$\Omega^-(Z) = -Q_{x_0}\left(z - \frac{K^+ - K^-}{K^+ + K^-}\frac{R^2}{z}\right) + \Phi_0 \qquad \text{for} \qquad z\overline{z} \geq R^2, \tag{14}$$

where Q_{x_0} is the uniform flow field intensity that is parallel to the x-axis and Φ_0 can be determined at a reference point with a known head.

The uniform flow shows the behavior of the streamlines in the domain. The definition of the uniform flow in the global coordinate system having intensity Q_{uf}, forming an angle α_{uf} relative to x-axis is given by:

$$\Omega_{uf} = -Q_{uf} z e^{-i\alpha_{uf}} + C, \tag{15}$$

and the complex potential for a well with intensity Q_w is given by:

$$\Omega_w = \frac{Q_w}{2\pi}\ln(z - z_w), \tag{16}$$

where C is called reference point constant and can be found at a point with known amount of hydraulic head, z represents any position and $z_w$represents the position of the well in the domain.

3. Solution algorithm

Boundary conditions have been attempted to be satisfied employing various strategies. The analytic elements result in linear systems, the solution to which has been developed by Strack and Jancović and Barnes, as an iterative method. The same methodology has been employed to solve circular inhomogeneities and other such problems. The coefficients to these circles, are found using the Eq. (12), expressed as:

$$a_{i,n} = P_{i,n}\int_0^{2\pi}\Re(\Omega_T(\theta_i))e^{-in\theta_i}d\theta_i, \tag{17}$$

or

$$a_{i,n} = P_{i,n} \int_0^{2\pi} \Phi_{\neq j}(\theta_i) e^{-in\theta_i} d\theta_i, \tag{18}$$

the expression determines the n^{th} coefficient for circle i, while the complex potential of all the elements is represented by Ω_T. The discharge potential $\Phi_{\neq j}$ represents the real part of the complex potentials of all other elements. The elements include the uniform regional flow as presented by Eq. (15). The set of linear equations generated by Eq. (17) are solved by either Iterative Method or Matrix Method.

3.1 Iterative Method

Considering a system of circular inhomogeneities with M circles in a uniform flow field and a well, the relation to determine the unknown coefficients for element i, employing Eq. (17), takes the form:

$$a_{i,n} = P_{i,n} \sum_{j\neq i}^{M} \int_0^{2\pi} \Re(\Omega_j(Z_j(\theta_i))) e^{-in\theta_i} d\theta_i + P_{i,n} \int_0^{2\pi} \Re(\Omega_{uf}(\theta_i) + \Omega_w(\theta_i)) e^{-in\theta_i} d\theta_i, \tag{19}$$

or

$$a_{i,n}^{k+1} = P_{i,n} \sum_{j\neq i}^{M} \int_0^{2\pi} \Re(\Omega_j(Z_j(\theta_i))) e^{-in\theta_i} d\theta_i + P_{i,n} \int_0^{2\pi} \Re(\Omega_{uf}(\theta_i) + \Omega_w(\theta_i)) e^{-in\theta_i} d\theta_i, \tag{20}$$

for which

$$P_{i,n} = \begin{cases} \frac{1}{2\pi}\left(\frac{K^+ - K^-}{K^-}\right) & for \quad n = 0 \\ \frac{1}{\pi}\left(\frac{K^+ - K^-}{K^+ + K^-}\right) & for \quad n > 0, \end{cases} \tag{21}$$

where i shows a particular element, while k represents the number of iterations. The regional uniform flow $\Omega_{uf}(\theta_i)$ and the complex potential $\Omega_w(\theta_i)$ of the well are as defined by Eq. (15) and Eq. (16) respectively.

3.2 Matrix Method

Despite its ease of understanding and implementation, the Iterative Method might occasionaly fail to converge. The counter to that problem necessitates the formation of a coefficient matrix **A** of the linear system

$$\mathbf{Ax} = \mathbf{b} \tag{22}$$

and solving it employing techniques suited for linear system. The matrix **A** is formulated from expanding power series of the potential of circular inhomogeneities. The uniform flow and the well contribute to the known vector **b** in Eq. (22).

Eq. (19) takes the form after simplification as:

$$\sum_{j\neq i}^{M} \sum_{m}^{N} \int_0^{2\pi} \Re(\Omega(Z_{j,i}(\theta_i)) e^{-in\theta_i} d\theta_i - \frac{a_{i,n}}{P_{i,n}} + \int_0^{2\pi} C e^{-in\theta_i} d\theta_i = - \int_0^{2\pi} \Re(\Omega_{uf} + \Omega_w) e^{-in\theta_i} d\theta_i, \tag{23}$$

and is used to calculate the unknown coefficients in the Matrix Method, where i is the number of elements, j is the specific element and m and n are order of the series.

4. Results

Initially, the uniform flow field influenced by singular circular inhomogeneity has been simulated. An analytical solution to this problem has been proposed by (Strack, 1989). The uniform flow comes out to be 0.5 m /day, while the aquifer happens to have background conductivity of 1 m/day. The circle of radius 50 meter demonstrates the hydraulic conductivity of 10 m/day and the numerical solution expand to the order of $N = 10$.

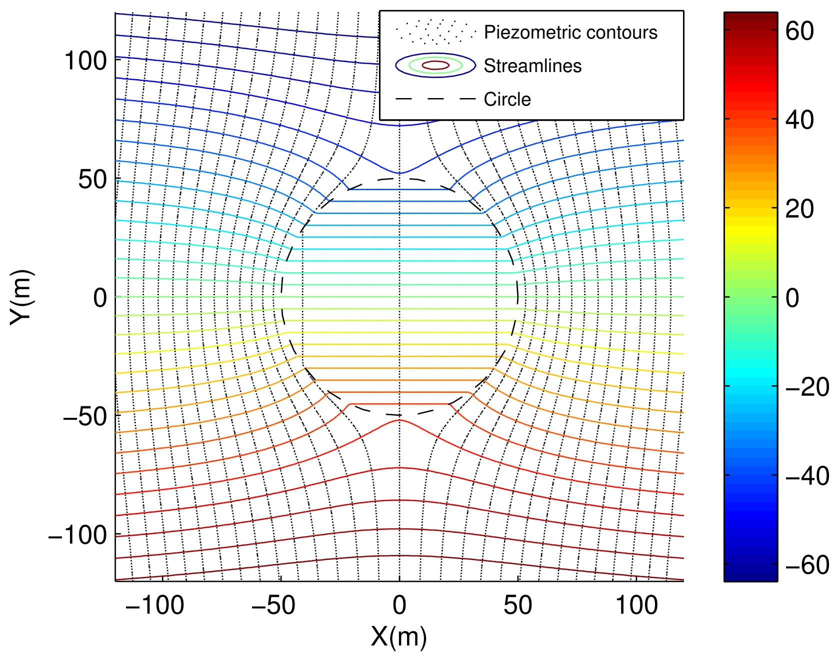

Figure 2. Analytic solution of a circular inhomogeneity in a uniform flow field with larger hydraulic conductivity than the background. Streamlines are horizontal and piezometric contours (hydraulic head) are in the vertical direction.

The analytical solution resulting in streamlines and hydraulic head are presented in Figure 2. The behaviours represented by analytical solution are confirmed by the numerical solution to the problem, determined by the Iterative and Matrix Method. By comparing the analytical solution with the numerical solution of the problem, we found that the maximal relative error is 9.5×10^{-17}, which is the order of machine epsilon.

In order to demonstrate the capabilities of our methods, further simulations were carried out for a more complex problem consisting of 10 circular inhomogeneities distributed randomly close to each other in a domain, with a uniform flow field and a well. The uniform flow intensity is $Q_{uf} = 0.5$ m/day along the horizontal direction as shown in Fig. 3, with parameters given in Tab. 1, and the well has discharge $Q_w = 100$ m^3/day is located at point z_w as is shown in Fig. 3. Despite being a challenging test case due to inhomogeneities in the proximity (4 and 7) and a well between them, the Matrix Method as well as the Iterative Method are able to solve the expected flow field adequately. The methods have the ability to simulate the flow field at multiple scales, as demonstrated by the results.

Table 1. Hydraulic conductivities, radii and series expansion order for the circles shown in Fig. 3.

Circle	K^+ m/d	Radius	Order
1	10	50	50
2	0.001	20	50
3	5	50	50
4	0.0008	30	50
5	0.0005	60	50
6	20	70	50
7	50	40	50
8	0.00007	70	50
9	15	60	50
10	0.0003	50	50

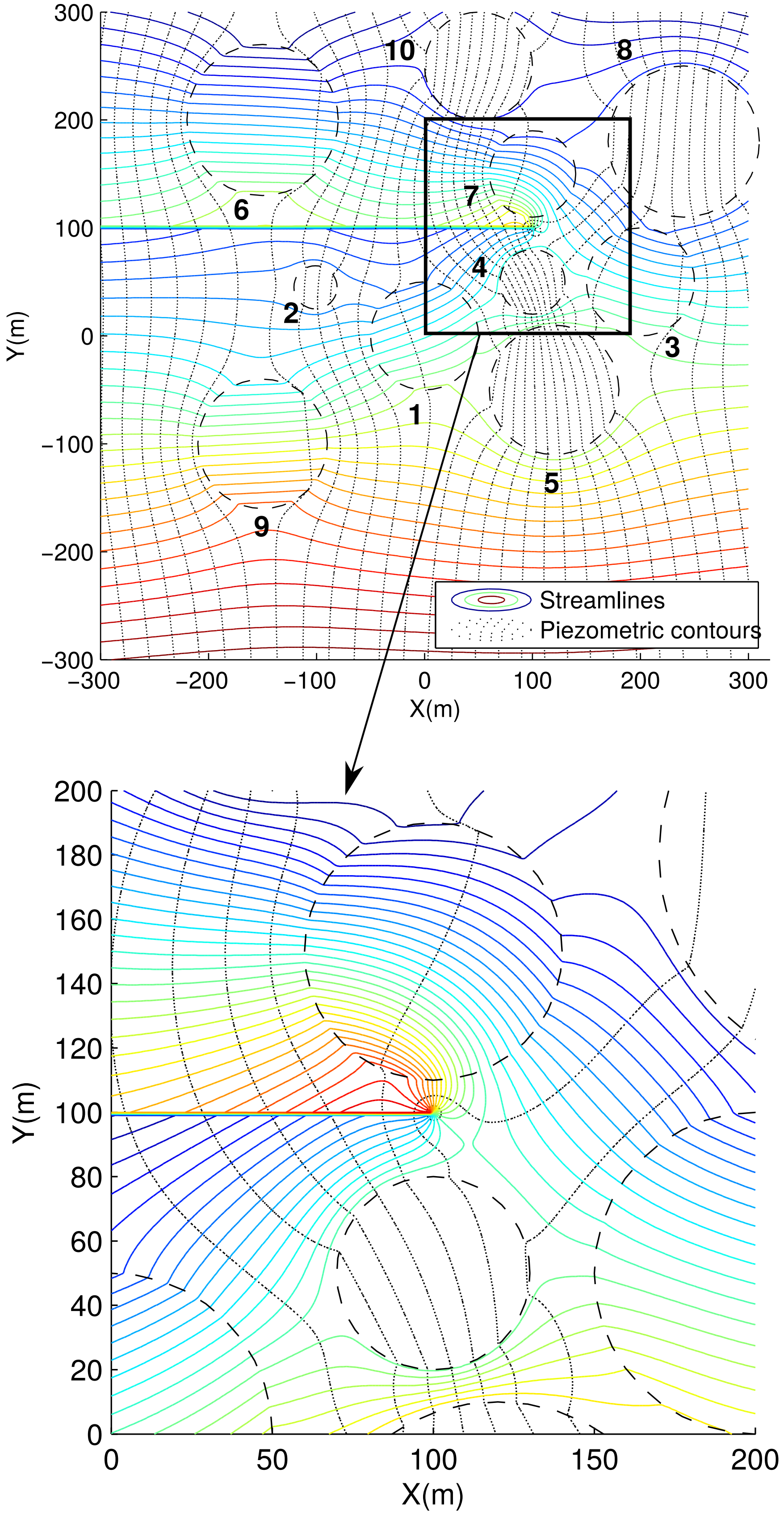

Figure 3. Streamlines and piezometric contours obtained from the simulation of 10 circular inhomogeneities in an uniform flow field and a well, with parameters given in Tab. 1.

The given results show the behaviour of the streamlines and piezometric contours and it is noticed that when the hydraulic conductivity inside the inhomogeneities is greater than the medium, the streamlines enter the inhomogeneities whereas the piezometric contours avoid to enter due to high permeability or vice versa. During simulations it is observed that the Iterative Method as well as the Matrix Method gives the same results in all the examined cases.

As illustrated, Fig. 4 shows the processing time for both the Iterative and the Matrix Method for the problem having 10 circular inhomogeneities (Fig. 3). The Matrix Method proved faster while simulating the problem, as compared to the Iterative Method, proving the Matrix Method to have lower computational cost. The Iterative Method needs a large number of steps in order to satisfy the criteria for convergence while the matrix solver constructs coefficients matrix as a first step, and then solve the system once.

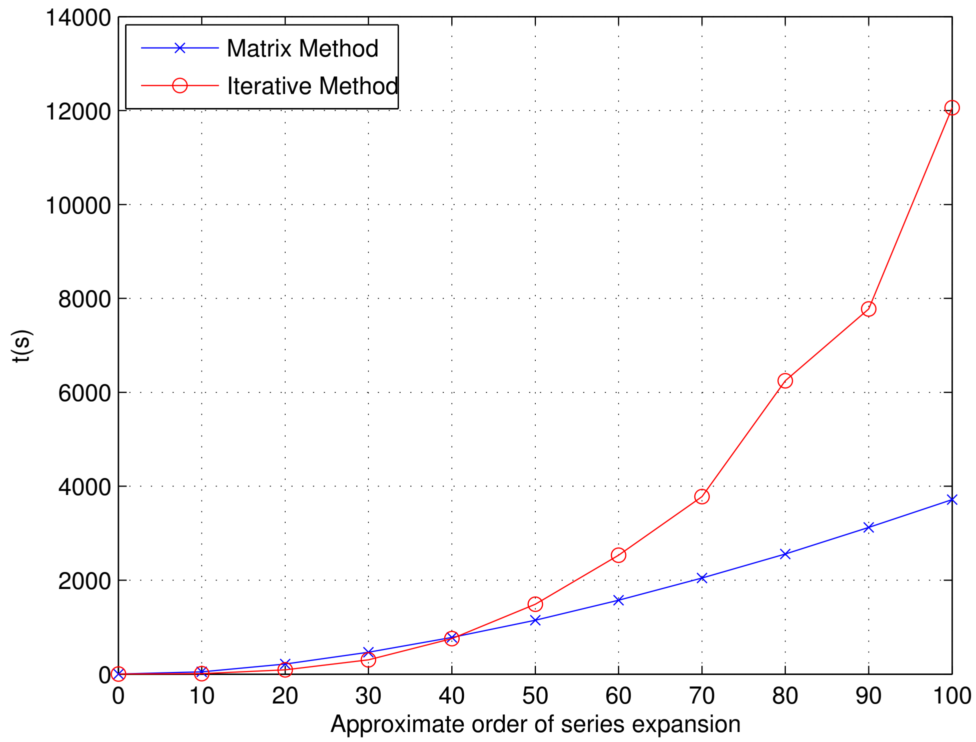

Figure 4. The processing times, as obtained from simulation of the system of circular inhomogeneities shown in Fig. 3, by both the Iterative and Matrix Method.

5. Conclusion

A matrix approach has been presented to determine the expansion coefficients for analytical elements like circles. Circular inhomogeneities were simulated using both the Iterative and the Matrix Method and almost identical results were obtained. The agreement of the numerical solution with the exact solution was excellent, with relative error smaller than machine precision. Moreover, it is common in numerical simulation and computation that there is a compromise between memory usage and processing time. The computational cost of the Iterative Method is higher in processing time and less in memory use, whereas in the Matrix Method the inverse occurs. Higher memory usage leads to lower processing costs, and vice versa. This relation was observed in the comparison of the solutions to the same problem by the Iterative and the Matrix Method.

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