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ANALITYCAL SOLUTION OF NONLINEAR TRANSIENT HEAT CONDUCTION PROBLEM USING GREEN'S FUNCTIONS

Sidney Ribeiro
Gabriela Costa
Federal University of Uberlandia
sidney.ribeiros@hotmail.com
gabriela_costadeoliveira@hotmail.com

Gilmar Guimaraes
Federal University of Uberlandia
gguima@ufu.br

Abstract. *Analytical solutions showed to be an important and strong tool for understand phsysical problems using mathematic tools. In this work we propose an approach about analytic solution for nonlinear transient heat conduction problems using mathematic elements such as Kirchhoff transformation and Green's functions. Using the combination of this methods we showed that was possible to determinate an analitycal solution for this thermal problem, and showed a good aproximation whem compared whith results from numerical methods.*

Keywords: *Nonlinear problems, heat conduction, Kirchhoff transformation, Green's functions*

1. INTRODUCTION

Analitycal solutions shows an important tool for develop solutions on engineering problems, once that analitycal solution can be used for validate numerical solutions or, aproximate solution, that makes it easy to analyze and understand physical problems (Fernandes, 2009). The complexity of thermal model from the point of view of obtaining analytical solutions, usually in transient multidimensional problems submitted to non-homogeneities such as prescribed flow boundary conditions, heat generation or boundary condictions as temperature varying over time.

Transient, multi-dimensional analitycal solutions whith heat generation or heat flux using Green's functions was showed by (Cole *et al.*, 2010). The text presents theoretical development and examples of multi-dimensional application, change of variables, the Green's function method with finite and semi-infinite geometries.

(Sun *et al.*, 2008) using the Kirchhoff integral transformation, showed a model to estimate levels of concrete cloride concentration exposed to a cloride environment. The uni-dimensional model was used, where the cloride coeficient therm was concrete concentration dependent. Thus, a model was linerized using Kirchhoff transformation and, using a linerized model it was possible to determine the analitycal solution, and after, using the inverse Kirchhoff transformation it was obtained the solution for non linear problem. But was consider a steady state heat conduction equation, it become easily solve this problem.

(Zhang *et al.*, 2013) presented the development of a solution of a nonlinear problem using Green's functions by the artificial parameter method, this method consist in replace the nonlinear terms such that: Thermal conductivity, density and specific heat, by an artificial parameter, and in this way linearizes the thermal model. Comparisons were made between the analytical solution of the problem considering temperature-dependent properties and constants in order to analyze the influence of non-linearity on the thermal behavior.

The new method to estimate thermal properties are temperature dependent was proposed by (Cui *et al.*, 2012), this method consist in use the inverse solution. Initially the thermal properties it is not required, the unknow values are treated as optimization variables, and the erros are minimized between experimental and calculed temperatures. The least squares method was used. Were shown examples of the method eficiency and several ways of use in engineering.

Thus, this paper propose through of mathematical analysis show the method to determine an analitycal solution of nonlinear transiente heat conduction equation using the Green's function method.

2. MATHEMATICAL MODEL

2.1 The uni-dimensional nonlinear X22 thermal model

Heat conduction problems can be described using a notation acording to their boundary condiction. This notation was proposed by (Cole *et al.*, 2010).

Thus we will use this notation that consists by letters X, Y and Z describe the axis directions and numbers 1, 2, 3 boundary condictions of first, second and third kinds respectively.

So the $X22$ thermal model describe an uni-dimensional model with bondary conditions of second kind in $x = 0$ and $x = L$ where $0 \leq x \leq L$ (Cole *et al.*, 2010).

The problem is described by nonlinear diferential equation:

$$k(T)\frac{\partial^2 T}{\partial x^2} = C(T)\frac{\partial T}{\partial t} \tag{1}$$

Note that the therms $k(T)$ and $C(T)$ are temperature dependent, that is, nonlinearites. Where $C(T) = \rho(T)C_p(T)$.

Subjected to the boundary conditions:

$$-k(T)\frac{dT}{dx}\Big|_{x=0} = q \tag{2}$$

and

$$-k(T)\frac{dT}{dx}\Big|_{x=L} = 0 \tag{3}$$

Note that the boundary condictions also have nonlinearities.

Under the initial condiction:

$$T(x,0) = T(x) \tag{4}$$

Normaly, all the analytic methods to solve a diferential partial equation needs to use the superposition theorem, but, in this case it is not possibile due the nonlinearities. Thus, it is necessary to determine a auxiliary problem.

2.2 The auxiliary thermal model problem

(Carslaw and Jaeger, 1959) defines Kirchhoff transformation as:

$$\Phi(x,t) = k(T) = \frac{1}{k_0}\int_0^T k(\widehat{T})d\widehat{T} \tag{5}$$

where k_0 is the reference thermal condutivity (Kevin R. Bagnall, 2014).

By Eq.(5) by integral definition, we have:

$$\frac{d\Phi}{dT} = \frac{k(T)}{k_0} \Rightarrow k(T) = k_0\frac{d\Phi}{dT} \tag{6}$$

Thus, by Eq. (1) we have:

$$\frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}\right) = C(T)\frac{\partial T}{\partial t} \tag{7}$$

Applying Eq.(6) in left side of Eq.(6):

$$\frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}\left(k_0\frac{d\Phi}{dT}\frac{\partial T}{\partial x}\right) \Rightarrow k_0\frac{\partial^2 \Phi}{\partial x^2} \tag{8}$$

Multiplying the rigth side of Eq. (7) by $\frac{\partial \Phi}{\partial \Phi}$ and using Eq.(6) we get:

$$C(T)\frac{\partial T}{\partial t}\frac{\partial \Phi}{\partial \Phi} \Rightarrow \frac{k_0}{\alpha(T)}\frac{\partial \Phi}{\partial t} \tag{9}$$

So, we have the auxiliary problem:

$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{1}{\alpha(T)}\frac{\partial \Phi}{\partial t} \tag{10}$$

Note that, in terms of Φ Eq. (10) remains nonlinear, but in a weaker form, doing $\alpha(T) = \alpha_0$, we have:

$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{1}{\alpha_0}\frac{\partial \Phi}{\partial t} \tag{11}$$

Eq.(11) define a linear auxiliary model of Eq.(1) model.
For boundary condictions we use Eq.(6) in the same way.

$$-k(T)\frac{dT}{dx}\Big|_{x=0} = q \Rightarrow k_0\frac{d\Phi}{dx}\frac{dT}{dT}\Big|_{x=0} = q \Rightarrow k_0\frac{d\Phi}{dx}\Big|_{x=0} = q \tag{12}$$

and

$$-k_0\frac{d\Phi}{dx}\Big|_{x=L} = 0 \tag{13}$$

and the initial condiction

$$\Phi(x,0) = \Phi(x) \tag{14}$$

Thus, we can use the Green's functions method for stabilish the $\Phi(x,t)$ solution.

2.3 The inverse Kirchhoff transformation

Starting from:

$$\Phi = k(T) = \frac{1}{k_0}\int_0^T k(\widehat{T})d\widehat{T} \tag{15}$$

considering $k(T) = k_0(1 + AT)$, where A is a constant, we have:

$$\Phi(T) = k(T) = \frac{1}{k_0}\int_0^T k_0(1 + A\widehat{T})d\widehat{T} \Rightarrow \Phi(T) = k(T) = \frac{AT^2}{2} + T \tag{16}$$

Thus

$$T(x,t) = k^{-1}(\Phi) \Rightarrow T(x,t) = \frac{1}{A}\left(\sqrt{1 + 2A\Phi(x,t)} - 1\right) \tag{17}$$

It is a inverse Kirchhoff transformation.

3. X22 NONLINEAR THERMAL PROBLEM

The solution of auxiliary problem using Green's functions is given by

$$\Phi(x,t) = \Phi_0 + \frac{\alpha_0}{k_0}\int_0^\tau G_{x22}(x,t|x',\tau)q d\tau \tag{18}$$

where $G_{x22}(x,t|x',\tau)$ it is the $X22$ Green's function (Cole *et al.*, 2010), it is given by

$$G_{x22}(x,t|x',\tau) = \frac{1}{L} + \frac{2}{L}\sum_0^\infty e^{-\beta_m^2\alpha_0(t-\tau)}cos(\beta_m x)cos(\beta_m x') \tag{19}$$

Thus

$$\Phi(x,t) = \Phi_0 + \frac{q\alpha_0 t}{k_0 L} + \frac{2q}{k_0 L}\sum_0^\infty \frac{cos(\beta_m x)}{\beta m^2} - \frac{2q}{k_0 L}\sum_0^\infty cos(\beta_m x)\frac{e^{-\beta_m^2\alpha_0 t}}{\beta_m^2} \tag{20}$$

where $\beta_m = \frac{m\pi}{L}$
Thus

$$T(x,t) = k^{-1}(\Phi) \Rightarrow T(x,t) = \frac{-1 + \sqrt{2A\Phi(x,t) + 1}}{k_0} \tag{21}$$

It is the nonlinear analitycal solution of $X22$ thermal problem.

Using: $L = 0,01m$, $x = 0m$, $T(x,0) = 0^oC$, $k(T) = k_0(1 + AT)$, $k_0 = 14,9W/mK$, $\alpha_0 = 3,95 \times 10^{-6}m^2/s$, $A = 1 \times 10^{-3}$ and $q = 1 \times 10^5W$ and software Scilab to compute Eq.(21) we have:

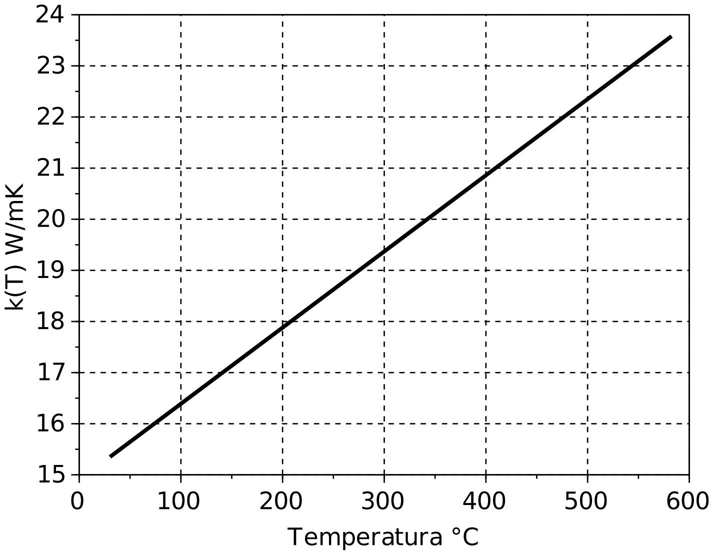

Figure 1. Behavior of $K(T)$ term in $600s$

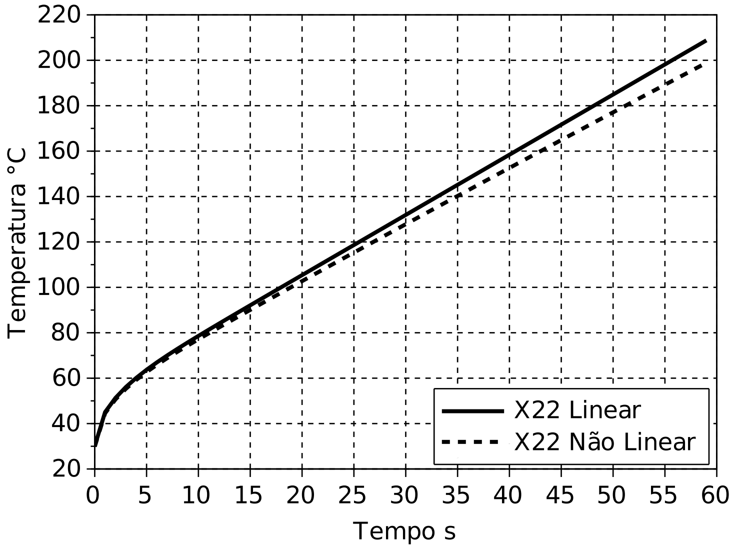

Figure 2. Comparison between X22 linear and X22 nonlinear

Fig.(2) shows a comparison of linear and nonlinear solution from X22 problem.

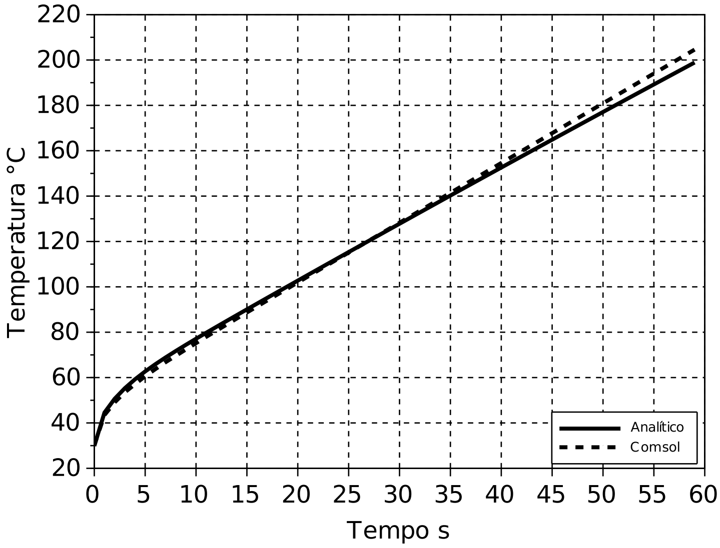

Figure 3. Comparison between X22 nonlinear and X22 numerical solution

Fig.(3) shows a comparison between X22 nonlinear analitycal solution and a numerical solution soved by COMSOL Multiphisycs.

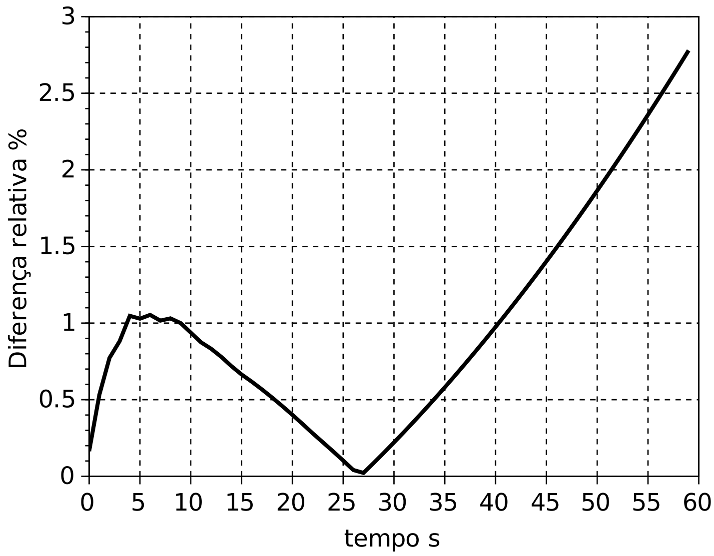

Figure 4. Relative diference

Fig.(4) shows that, the relative diverence between numrical and analitycal solutions are less than 2.7% that is a good aproximation.

4. CONCLUSION

This work proposed an way to solve the transient heat conduction using Green's functions. Was showed the mathematical analisys and Kirchhoff variables changes to obtain a linear model and boundary condictions version, and so it was possible to aply the Green's function method due the possibility of use the superposition theorem

After returning to the original problem using the inverse Kirchhoff transformation it was possible to show a graphic problem solution and make comparisons with numerical solution.

The method presented a good aproximation when compared to the numerical solution, where all differences less than 2.7%.

5. ACKNOWLEDGEMENTS

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