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INVERSE PROBLEM OF A ONE-DIMENSIONAL MODEL IN MULTILAYER HEAT CONDUCTION

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Abstract. *The inverse problem in conducting heat is related to the determination of the boundary condition, rate of heat generation, or thermophysical properties, using temperature measurements at one or more positions of the solid. The inverse problem in conducting heat is mathematically one of the ill-posed problems, because its solution is extremely sensitive to measurement errors. For a well-placed problem the following conditions must be satisfied: the solution must exist, it must be unique and must be stable on small changes of the input data. The objective of the work is to estimate the heat flux generated at the tool-chip-chip interface in a manufacturing process. The term "estimation" is used because in the temperature measurements, errors are always present and these affect the accuracy of the calculation of the heat flow.*

Keywords: *inverse problem, green function, heat conduction*

1. INTRODUCTION

In the machining process, the use of inverse problems is common for the estimation of the temperature at which the tip of the cutting tool is located. The direct measurement of sensor temperature becomes complicated due to the rotating movement of the tool, the use of thermocouples, and the release of chips, which makes the measurement by radiation. The temperature of the cutting tool influences its useful life and a better control of machining parameters such as feed and cut speeds, enables increased productivity and lower production costs.

Several techniques are used in the inverse problem solutions in heat conduction, such as: Least squares method, Duhamel's theorem, Specified function, genetic algorithm, simulated annealing, more recently we have used filters for solution of inverse problem in conduction of heat. Another technique has been proposed and developed by (FERNANDES, 2013) that relates inverse problem with transfer function TFBGF (transfer function based green's function).

The methodology of the technique (TFBGF) consists in determining the transfer function of a thermal system. With the transfer function it is possible to establish the method of estimation of the heat flux by different approaches, through the deconvolution. That is, with the knowledge of the transfer function (H (x, t)) and the hypothetical or simulated temperature (T (x, t)) it is possible to estimate the heat flux $q(s) = T(x,s)\frac{1}{H(x,s)}$, given in the domain of Laplace (FERNANDES, 2013).

2. MATERIALS AND METHODS

In this section we address the direct and inverse problem in multilayer heat conduction.

2.1 Direct multilayer problem

The thermal problem is defined by a flat plate, with two layers, subjected to a heat flow, q(t) in $x = 0$, and thermal insulation condition on the opposite surface, $x = L$, whose thermophysical properties are different in each layer, delimited by $x = b$ Fig. (1), is referenced as $X2C12$ por (Haji-Sheikh, 2014).

The analytical solution of the problem represented by Fig. (1) is developed in (Oliveira, 2015) and it has as analytical solution:

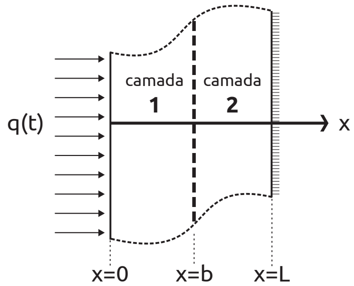


Figure 1. Thermal problem: flat plate, with two layers, subjected to a heat flux in $x = 0$, and thermal insulation condition on the opposite surface, $x = L$.

$$\begin{aligned} T_1(x,t) &= \frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{X_{1n}}{N_x}\int_0^t e^{-\lambda_n(t-\tau)}\int_{x_1}^{x_2} X_{1n}(x')q(t)\delta(x'-0)dx'd\tau \\ &= \frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{X_{1n}(x)X_{1n}(0)}{N_x}\int_0^t q(t)e^{-\lambda_n(t-\tau)}d\tau \\ &= \frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{cos(\gamma x)cos(0)}{N_x}\int_0^t q(t)e^{-\lambda_n(t-\tau)}d\tau \end{aligned} \tag{1a}$$

$$\begin{aligned} T_2(x,t) &= \frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{X_{2n}}{N_x}\int_0^t e^{-\lambda_n(t-\tau)}\int_{x_1}^{x_2} X_{1n}(x')q(t)\delta(x'-0)dx'd\tau \\ &= \frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{X_{2n}(x)X_{1n}(0)}{N_x}\int_0^t q(t)e^{-\lambda_n(t-\tau)}d\tau \\ &= \frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{1}{N_x}\left\{\left[cos(\eta b)cos(\gamma b)+\left(\frac{k_1}{k_2}\right)\left(\frac{\gamma}{\eta}\right)sen(\gamma b)sen(\eta b)\right]cos(\eta x)\right. \\ &+\left.\left[cos(\gamma b)sen(\eta b)-\left(\frac{k_1}{k_2}\right)\left(\frac{\gamma}{\eta}\right)sen(\gamma b)cos(\eta b)\right]sen(\eta x)\right\}cos(0) \\ &\times\int_0^t q(t)e^{-\lambda_n(t-\tau)}d\tau \end{aligned} \tag{1b}$$

Where λ_n are n eigenvalues and γ and η are related to λ by the following expression

$$\gamma^2 = \frac{\lambda^2}{\alpha_1} \qquad e \qquad \eta^2 = \frac{\lambda^2}{\alpha_2} \tag{2}$$

It is observed that to obtain the thermal profile of the problem $X2C12$ a particular case is used, where the heat flux imposed at one end ($x = 0$) is known and is costly, leaving only the calculation of the integrals in the Eqs. (1). For example, if the heat flux is constant or only dependent on the position, or even an exponential function, $q(t) = c_1e^{-c_2t}$, with c_1 and c_2 nonzero, the solution is easily determined analytically. However, in a real situation the heat flux, $q(t)$, is not described by an analytical expression, since its nature is discrete. In this case, the solution could be called a "hybrid", because the integral is necessarily calculated from the heat flow discretization, Fernandes (2013). The hybrid solution that is used in a real situation is shown below, since the heat flux is discrete.

2.2 Hybrid solution

The hybrid solution is an alternative to real cases where the heat flux is not constant. In this case, the experimental heat flux (discrete data) is represented as a vector where each component is a flux value, and this flux is considered constant for each time interval, as shown in the figure, that is, $q(t) = [q_1, q_2, q_3, \cdots, q_n]$ being q_n the component for the interval $\triangle_t = t_{n+1} - t_n$ with $n = 1, 2, 1$ Fernandes (2013). Thus, the integral appearing in Eqs. (1a) - (**??**) can be expressed by

$$\begin{aligned} \int_0^t q(t)e^{-\lambda_n^2(t-\tau)}d\tau &= \int_{t_1=0}^{t_2} q_1e^{-\lambda_n^2(t-\tau)}d\tau + \int_{t_2}^{t_3} q_2e^{-\lambda_n^2(t-\tau)}d\tau + \cdots \\ &+ \int_{t_n}^{t_{n+1}} q_ne^{-\lambda_n^2(t-\tau)}d\tau = \frac{1}{-\lambda_n^2 t}\sum_{n=1}^{N-1} q_n(e^{-\lambda_n^2(t_n+1)} - e^{-\lambda_n^2(t_n)}) \end{aligned} \tag{3}$$

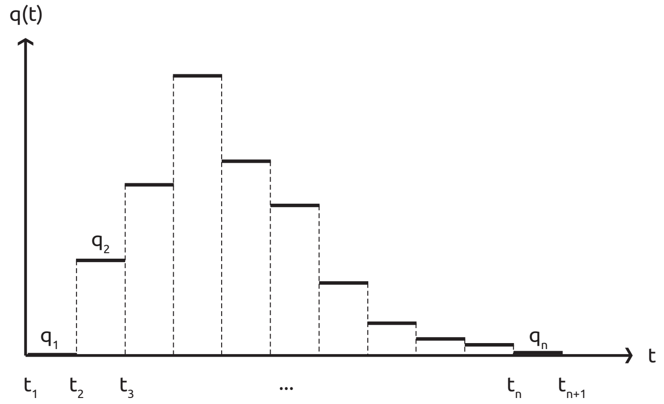


Figure 2. Discrete heat flow.

Then, the temperature solution given by Eqs. (1a) - (1b) can be rewritten as follows

$$T_1(x,t) = \frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{cos(\gamma x)cos(0)}{N_x}\frac{1}{-\lambda_n^2 t}\sum_{n=1}^{N-1} q_n(e^{-\lambda_n^2(t_n+1)} - e^{-\lambda_n^2(t_n)}) \quad (4a)$$

$$\begin{aligned} T_2(x,t) = {} & \frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{1}{N_x}\left\{\left[cos(\eta b)cos(\gamma b) + \left(\frac{k_1}{k_2}\right)\left(\frac{\gamma}{\eta}\right)sen(\gamma b)sen(\eta b)\right]cos(\eta x)\right. \\ & + \left.\left[cos(\gamma b)sen(\eta b) - \left(\frac{k_1}{k_2}\right)\left(\frac{\gamma}{\eta}\right)sen(\gamma b)cos(\eta b)\right]sen(\eta x)\right\}cos(0) \\ & \times \frac{1}{-\lambda_n^2 t}\sum_{n=1}^{N-1} q_n(e^{-\lambda_n^2(t_n+1)} - e^{-\lambda_n^2(t_n)}) \end{aligned} \quad (4b)$$

In order to validate the hybrid solution given by Eq. (4), the analytical solution is obtained by considering the heat flux, $q(t) = c_1 e^{-c_2 t}$ in Eq. (1) and thus, solving the integrals in the time of the Eqs. (1a) - (1b)

$$T_1(x,t) = \frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{cos(\gamma x)cos(0)}{N_x}\frac{c_1(e^{-c_2 t} - e^{-\lambda_n^2 t})}{\lambda_n^2 - c_2} \quad (5a)$$

$$\begin{aligned} T_2(x,t) = {} & \frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{1}{N_x}\left\{\left[cos(\eta b)cos(\gamma b) + \left(\frac{k_1}{k_2}\right)\left(\frac{\gamma}{\eta}\right)sen(\gamma b)sen(\eta b)\right]cos(\eta x)\right. \\ & + \left.\left[cos(\gamma b)sen(\eta b) - \left(\frac{k_1}{k_2}\right)\left(\frac{\gamma}{\eta}\right)sen(\gamma b)cos(\eta b)\right]sen(\eta x)\right\}cos(0) \\ & \times \frac{c_1(e^{-c_2 t} - e^{-\lambda_n^2 t})}{\lambda_n^2 - c_2} \end{aligned} \quad (5b)$$

The solutions given by Eqs. (4)- (5) are implemented in MATLAB with the following physical and geometric characteristics, thermal conductivity, $k_1 = 401[W/mK]$, $k_2 = 401[W/mK]$, thermal diffusivity, $\alpha_1 = 117E-6[m^2/s]$, initial temperature $T_0 = 0[C]$, plate length, $L = 5E-2[m]$ and e $dt = 1; 0.5; 0.1; 0.01[s]$. For the discrete heat flux, applied only in Eq.(4), we have the vector $q_n = [0\ c_1 exp(-c_2 t)]$, with $c_1 = 4E5$ e $c_2 = 0.002$.

The table (1) shows the comparison between purely analytical (Eq. (5)) and hybrid solutions (Eq. (4)).

It is observed that the smaller the time discretization interval the closer the solutions will be to each other.

2.3 Transfer Function and Inverse multilayer problem

In dynamic systems there are three variables to be studied, the excitation, the transfer function and the response of the system, in this way the problems are solved knowing always two variables and estimating the third one: the inverse problems are those in which from the knowledge of the dystheme and its response (effect) the excitation (cause) is estimated. It is possible to analyze heat conduction problems by making an analogy to dynamic systems (Fernandes, 2013).

To obtain the analytical transfer function propose a methodology based on the theory of dynamic systems of an input and an output. That is, for any input $x(t)$, the output $y(t)$ is given by the convolution integral.

Table 1. Comparison between the solutions obtained by Eqs. (4) and (5)

$x = 0$	$dt = 1s$			$dt = 0,5s$		$dt = 0,1s$		$dt = 0,01s$	
t	Eq. (4)	Eq. (5)	dif.	Eq. (5)	dif.	Eq. (5)	dif.	Eq. (5)	dif.
$0,0$	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
$1,0$	$4,0033$	$4,0513$	$0,0480$	$4,0066$	$0,0033$	$4,0050$	$0,0017$	$4,0039$	$0,0006$
$2,0$	$6,6592$	$6,7533$	$0,0941$	$6,6648$	$0,0056$	$6,6622$	$0,0030$	$6,6593$	$0,0001$
$3,0$	$9,0282$	$9,1498$	$0,01216$	$9,0359$	$0,0077$	$9,0233$	$0,0049$	$9,0323$	$0,0041$
$4,0$	$11,3479$	$11,4883$	$0,1404$	$11,3577$	$0,0098$	$11,3415$	$0,0064$	$11,3533$	$0,0054$
$5,0$	$13,6561$	$13,8122$	$0,1561$	$13,6679$	$0,0118$	$13,6639$	$0,0078$	$13,6626$	$0,0065$
$6,0$	$15,9585$	$15,1338$	$0,1753$	$15,9724$	$0,0139$	$15,9499$	$0,0086$	$15,9661$	$0,0076$
$7,0$	$18,2561$	$18,4578$	$0,2017$	$18,2722$	$0,0161$	$18,2658$	$0,0097$	$18,2649$	$0,0088$
$8,0$	$20,5492$	$20,7726$	$0,2234$	$20,5673$	$0,0181$	$20,5611$	$0,0119$	$20,5591$	$0,0099$
$9,0$	$22,8376$	$23,0789$	$0,2413$	$22,8578$	$0,0202$	$22,8511$	$0,0135$	$22,8487$	$0,0111$
$10,0$	$24,1213$	$24,3819$	$0,2606$	$24,2231$	$0,1018$	$24,1356$	$0,0143$	$24,1103$	$0,0110$

$$f(t) = (g * q)(t) = \int_{-\infty}^{\infty} q(\tau)g(t-\tau)d\tau \tag{6}$$

Mathematically, in terms of heat flow q (input) and temperature T (response), we have

$$T_1(x,t) = h_1(x,t) * q(t) = \int_0^t h_1(x,t-\tau)q(\tau)d\tau \tag{7}$$

$$T_2(x,t) = h_2(x,t) * q(t) = \int_0^t h(x,t-\tau)q(\tau)d\tau \tag{8}$$

where the function $h(x,t) = 0$ for $\tau < 0$.

Knowing then the FG that characterizes the problem is possible to identify the impulsive response of the system, and therefore, its transfer function. The transfer function of a system is defined as the Laplace transform of the system's impulsive response.

Thus, since the FG of the problem $X2C12$ whose solution is given by the equations (1) is known, and using the Convolution theorem given by Eq. (6) we have

$$\Theta_1(x,t) = T_1(x,t) = \frac{\alpha_1}{k_1}\int_0^t G_{11}(x,t|0,t-\tau)q(t)d\tau \tag{9}$$

$$\Theta_2(x,t) = T_2(x,t) = \frac{\alpha_1}{k_1}\int_0^t G_{21}(x,t|0,t-\tau)q(t)d\tau \tag{10}$$

Substituting the FG that characterizes the $X2C12$ problem and rearranging the terms of the Eqs. (11) - (10)

$$\Theta_1(x,t) = T_1(x,t) = \frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{1}{N_x}cos(\gamma x) \times \int_0^t q(t)e^{-\lambda_n(t-\tau)}d\tau \tag{11}$$

$$\begin{aligned}\Theta_2(x,t) = T_2(x,t) = \frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{1}{N_x}\Bigg\{&\left[cos(\eta b)cos(\gamma b) + \left(\frac{k_1}{k_2}\right)\left(\frac{\gamma}{\eta}\right)sen(\gamma b)sen(\eta b)\right]cos(\eta x)\\ &+ \left[cos(\gamma b)sen(\eta b) - \left(\frac{k_1}{k_2}\right)\left(\frac{\gamma}{\eta}\right)sen(\gamma b)cos(\eta b)\right]sen(\eta x)\Bigg\}\\ &\times \int_0^t q(t)e^{-\lambda_n(t-\tau)}d\tau\end{aligned} \tag{12}$$

It is observed that the Eq. (16) is given by the product of terms that are dependent on τ and the delay $t-\tau$, which characterizes Convolution Theorem Eqs. (7) - (8). Since the transfer function is independent of input and response, the Dirac Delta function, $q(t) = (t)$, is proposed as input signal (heat flux), so Eqs. (7) - (8) become as follows:

$$T_1(x,t) = h_1(x,t) * \delta(t) = \int_0^t h(x,t-\tau)\delta(\tau)d\tau = h_1(x,t) \tag{13}$$

$$T_2(x,t) = h_2(x,t) * \delta(t) = \int_0^t h(x,t-\tau)\delta(\tau)d\tau = h_2(x,t) \tag{14}$$

Note that from the neutral element property of the convolution, that $h * \delta = h$, thus, we get the impulsive response without the need to solve the integral. Like this,

$$h_1(x,t) = \frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{1}{N_x}cos(\gamma x) \times \int_0^t \delta(\tau)e^{-\lambda_n(t-\tau)}d\tau \tag{15}$$

$$\begin{aligned} h_2(x,t) = \frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{1}{N_x}\Bigg\{&\left[cos(\eta b)cos(\gamma b) + \left(\frac{k_1}{k_2}\right)\left(\frac{\gamma}{\eta}\right)sen(\gamma b)sen(\eta b)\right]cos(\eta x) \\ &+ \left[cos(\gamma b)sen(\eta b) - \left(\frac{k_1}{k_2}\right)\left(\frac{\gamma}{\eta}\right)sen(\gamma b)cos(\eta b)\right]sen(\eta x)\Bigg\} \\ &\times \int_0^t \delta(\tau)e^{-\lambda_n(t-\tau)}d\tau \end{aligned} \tag{16}$$

Given the transfer function of the problem $X2C12$, by means of the Laplace transform of the impulse response, and as the knowledge of the system response (experimental or hypothetical temperature) it is possible to obtain an estimate for the excitation of the system, is, the flow of heat. This is the proposed procedure for the solution of the inverse problem that will be presented next.

It is known that, for a linear dynamic system, the relation between input and output in the domain of the complex variable s is given by the multiplication expressed in Eq. (4), or in the domain of time by convolution.

$$Y(s) = H(s) \cdot X(s) \tag{17}$$

Thus, the analysis of dynamic systems is facilitated by the use of the Laplace transform because it provides the mathematical relationship between the input and output of the dynamic system. In terms of the heat/temperature couple pair

$$\mathcal{L}[T_2(x,t)] = \mathcal{L}[h(x,t) * q(t)] \Rightarrow T_2(x,s) = H(x,s)\cdot q(s) \tag{18}$$

Note that Eq. (6) is a multiplication, so one can write

$$q(s) = \frac{1}{H(x,s)}\cdot T(x,s) \tag{19}$$

Also, in the time domain, the inverse Laplace transform is applied, and the convolution is obtained, in mathematical terms we have

$$\mathcal{L}^{-1}[q(s)] = \mathcal{L}^{-1}\left[\frac{1}{H(x,s)\cdot T(x,s)}\right] \Rightarrow q(t) = \frac{1}{h(x,t)} * T_2(x,t) \tag{20}$$

Numerically, this operation described by Eq. (7) will be performed on the MATLAB software using the following functions: *fft* and *ifft* which respectively apply the fast Fourier transform and its inverse to the functions T, q and h. The fast Fourier transform is used in this case, since it is known that the Laplace transform is a particular case of the Fourier transform.

3. RESULTS AND DISCUSSIONS

To obtain the solution of the one-dimensional direct problem, $X2C12$, given by the Eq. (4), called a hybrid, it is necessary to consider a discrete heat flux, $q = [q_1\ q_2\ cdots\ q_n]$. In this way, we construct it in the form of a vector, in MATLAB, as a triangular pulse, by means of the function *tripuls* ($q = c_1 * tripuls(t-c_2, c_2)$). The physical and geometric characteristics, thermal conductivity, $k_1 = 401[W/mK]$,, $k_2 = 401[W/mK]$, thermal diffusivity, $\alpha_1 = 117E^{-6}[m^2/s]$, $\alpha_2 = 117E^{-6}[m^2/s]$, temperature initial $T_0 = 0[C]$, plate length $L = 10E^{-2}[m]$ and $dt = 1[s]$, $t = [0 : dt : tf]$, $tf = 1024[s]$, constants c_1 and c_2 can assume any value, among them were chosen: $c_1 = 3 \times 10^5$ and $c_2 = 300$. Fig. (3) shows the triangular heat flux. The temperature distribution is obtained for the positions $x = 0$, b and L as shown in Fig.(4), in this way, by knowing the heat flow, the direct problem is established.

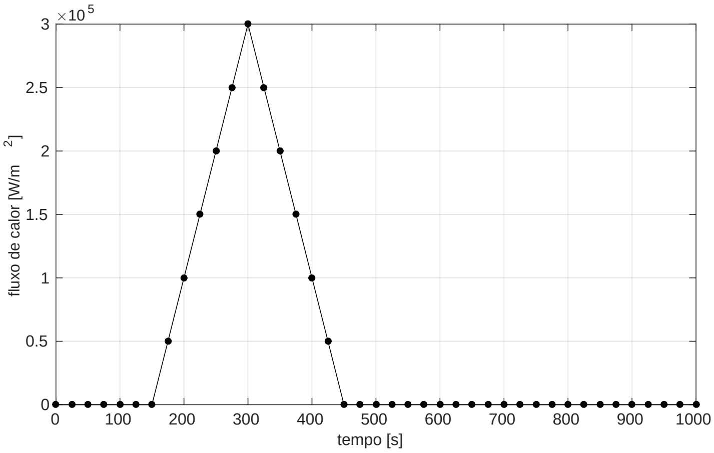


Figure 3. Fluxo de calor com pulso triangular.

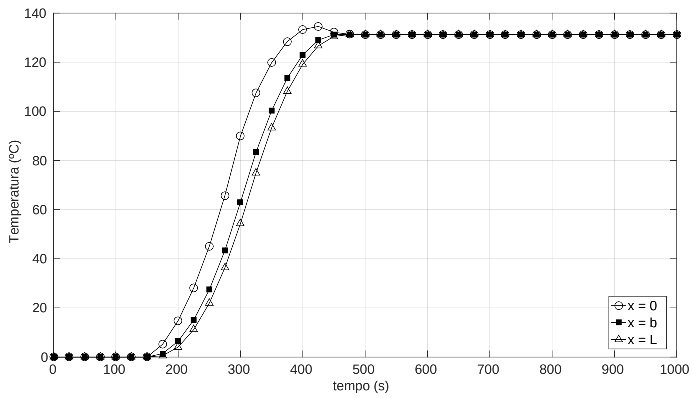


Figure 4. Temperaturas obtidas com fluxo de calor triangular.

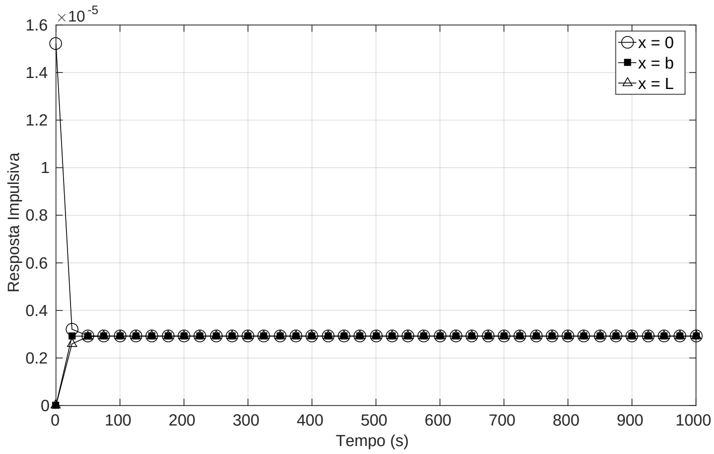
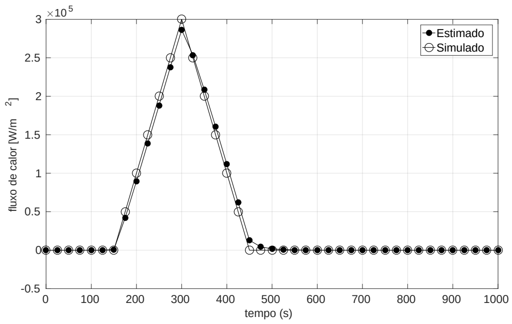


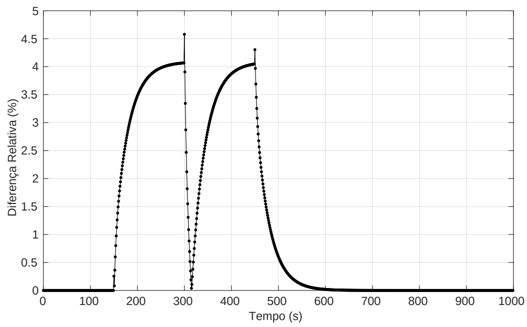
Figure 5. Impulse response.

The impulse response, is shown for the desired positions, that is, $x = 0, b, L$, considering the same data used previously to represent the attainment of the temperatures, thermo-physical properties, α_1, α_2, k_1, and k_2; length L, dt and df. Under these conditions the impulse response, h, presents the following graphical representation:

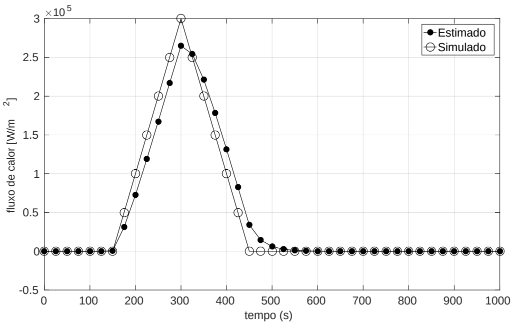
Note that in Fig. (5), the impulse response has a decreasing behavior from its maximum value. We have the following behavior of the heat flow term at the position $x = 0$, b and L.



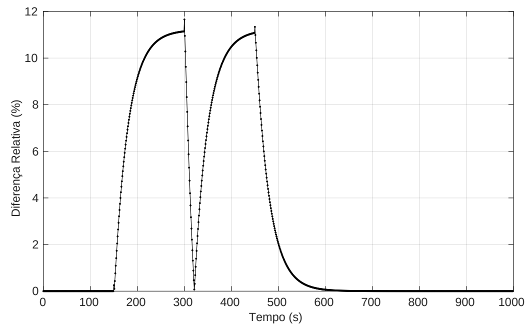
(a) Fluxo de calor estimado por temperatura e funÃğÃčo de transferÃłncia calculadas em $x = 0$.



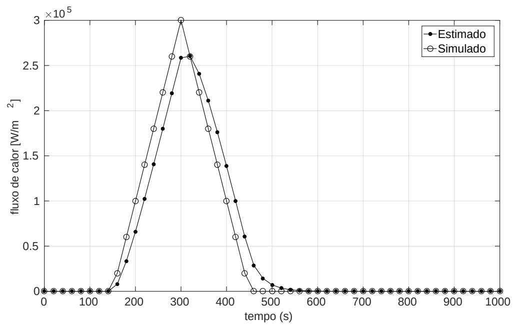
(b) DiferenÃğa Relativa entre o fluxo estimado e simulado $x = 0$.



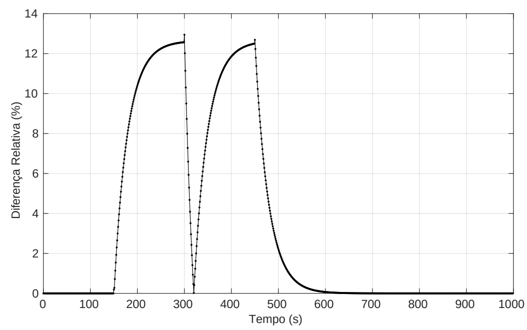
(c) Fluxo de calor estimado por temperatura e funÃğÃčo de transferÃłncia calculadas em $x = b$.



(d) DiferenÃğa Relativa entre o fluxo estimado e simulado $x = b$.



(e) Fluxo de calor estimado por temperatura e funÃğÃčo de transferÃłncia calculadas em $x = b$.



(f) DiferenÃğa Relativa entre o fluxo estimado e simulado $x = b$.

4. CONCLUSION

It can be seen in Figs, the position that had a best estimate was $x = 0$, in percentage terms the average error is of $4,5\%$ to the $x = b$ position of 11.7%, and the worst estimate was for $x = L$, position opposite the heat source, in this case the average error is 13%.

5. ACKNOWLEDGEMENTS

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