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# A NEARLY-ORTHOGONAL GRID GENERATION METHOD BASED UPON INHOMOGENOUS ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

**Kevin Eduardo de Conde**

**Ezio Castejon Garcia**

Instituto Tecnológico de Aeronáutica, 12228-900, São José dos Campos, São Paulo,  
kevin@ita.br, ezio@ita.br

**João Luiz F. Azevedo**

Instituto de Aeronáutica e Espaço, 12228-904, São José dos Campos, São Paulo,  
joaoluiz.azevedo@gmail.com

**Abstract.** *A nearly-orthogonal boundary-conforming grid generation technique based upon inhomogeneous elliptic partial differential equations is presented. The method has been designed to generate grids for multiply-connected domains in a block-structure fashion. It also allows for control of point distribution along boundaries, as well as control of grid spacing inside the domain by using a slightly modified version of the renowned Thompson's control functions, which is reported to work well. Although not fully-automatic, the process is automatized in order to make it more usable and speed-up computational time, which is usually a demotivating factor in elliptic grid generation techniques. This is also the reason why the implementation has been performed in an objected-oriented fashion using C++ and as relaxation method the Alternating Direction Implicit (ADI) with  $\alpha$ -sequence, generating appealing results. The method presented can be fully extended to 3D. Some results are presented and discussed.*

**Keywords:** *Boundary-conforming coordinate systems; Inhomogeneous elliptic partial differential equations; Nearly-orthogonal grid.*

## 1. INTRODUCTION

Grid generation can be understood as the process to obtain an adequate collection of points at which a system of equations is to be discretized and solved. In this context, an "optimum" point collection is desirable such that it minimizes the error in the solution. One possible way to obtain an optimized arrangement of points is by the use of boundary conforming coordinate systems, which is the reason why they have become very popular. Boundary-conforming coordinate systems are achieved by determining the values of the coordinates inside the physical region from the values and/or slopes of such coordinate lines on the boundaries. Classical approaches, such as algebraic interpolation and conformal mapping, are potential candidates, specially considering that such methods usually lead to great simplicity of implementation and low computational cost. Regarding algebraic interpolation methods, it has been pointed out that such methods may penalize applications for complex geometries, for example, collapsing lines of the same curvilinear coordinate (Ryskin and Leal, 1983); "transfinite interpolation methods" have become popular as initial condition for more sophisticated generators, as reported in Thompson *et al.* (1982), Eça (1996), among others. The foundations of algebraic interpolation methods in grid generation may be found in Gordon (1971) and Gordon and Hall (1973). Conformal mapping has also been widely used as a grid generation technique in early aircraft design, and references may be found in Fornberg (1980). As pointed by Ryskin and Leal (1983), conformal mappings are ill-conditioned in the sense that very small changes in the shape of the domain can dramatically alter the position of mapped boundary points. On the other hand, in conjunction with a partial differential equation, such problem configures a boundary value problem, and obtaining the grid points as the solution of such a problem is a viable alternative (Steger and Sorenson, 1979). In fact, the smoothness inherent in harmonic functions, and the uniqueness of the solution guaranteed by the maximum principles, tend to force that discontinuities and/or strong slopes present in the boundaries do not propagate inside the domain, what is desired for diminishing truncation errors in finite difference approximation for derivatives, as shown by Thompson *et al.* (1985). The authors still emphasize that, although the extremum principles may be weakened, or may even not be achieved by using more general elliptic systems (inhomogeneous systems), the conditions necessary for the extremum principles consist on a set of sufficient, but not necessary, conditions for a one-to-one mapping. Thus, as systems in the absence of control functions will originate grids that are not in general orthogonal to the boundaries, as well as with no control over grid spacing, it becomes more

convenient to use inhomogeneous elliptic generators, such as

$$\nabla^2 \zeta = \vec{Z}(\zeta), \quad (1)$$

where  $\zeta = (\xi, \eta)^t$ ,  $\mathbf{Z} = (P, Q)^t$ . Equation (1) corresponds to the following system in the computational domain

$$a_{22}\mathbf{r}_{\xi\xi} - 2a_{12}\mathbf{r}_{\xi\eta} + a_{11}\mathbf{r}_{\eta\eta} + J^2(P\mathbf{r}_{\xi} + Q\mathbf{r}_{\eta}) = 0, \quad (2)$$

where  $\mathbf{r} = (x, y)^t$ . The coefficients  $a_{11}$ ,  $a_{12}$  and  $a_{22}$  in Eq. (2) are called the covariant metric coefficients, and are defined by  $a_{11} = \mathbf{r}_{\xi} \cdot \mathbf{r}_{\xi}$ ,  $a_{12} = \mathbf{r}_{\xi} \cdot \mathbf{r}_{\eta}$ , and  $a_{22} = \mathbf{r}_{\eta} \cdot \mathbf{r}_{\eta}$ ;  $J$  is the Jacobian of the coordinate transformation, and naturally corresponds to  $J^2 = a_{11}a_{22} - (a_{12})^2$ . Such system was originally proposed by Thompson *et al.* (1974), reason why it is known as *TTM method*. Finally, it is important to emphasize that, although some might consider grid generation a solved issue, this is not really true for some specialized applications (Zhang *et al.*, 2008). An example is the case of multiply-connected domains with heterogeneous media; this is accentuated by fact that recent works have been addressing the subject (Zhang *et al.*, 2008; Wiesenberger *et al.*, 2017). This is the primary reason for the interest in the present work.

As pointed by Gordon and Hall (1973), since mesh generation usually involves both geometry and physics, and “automatic” grid generation methods consider only geometric aspects, we refrain from calling “automatic” such methods since the actual final mesh will ultimately be resolved by the analyst who is cognizant of its specific application.

## 2. METHODOLOGY

### 2.1 Contour Orthogonality

In the solution of problems of interest in computational fluid dynamics, orthogonality or near-orthogonality next to boundaries is usually desirable, since according to Wiesenberger *et al.* (2017), the accuracy on the discretization and boundary condition imposition is greatly improved. In case of a boundary corresponding to a wall, orthogonality and reduced grid spacing next to the wall are necessary in order to diminish the errors in the calculation of the large gradient of properties present in the boundary layer. Moreover, orthogonal or near-orthogonal grids allow the implementation of algebraic turbulence models to become more reliable, once the information normal to the boundary is usually needed in such models.

According to Khaymaseh *et al.* (1999), boundary orthogonality can be effected in two ways. In *Dirichlet-Neumann orthogonality*, control functions are not necessary. In this case, the points along the boundary are allowed to slide until boundary orthogonality is achieved, and the elliptic system has iterated to convergence. This approach is typically recommended for non-physical contours, such as far field boundaries, once it usually modifies the grid spacing near the boundary. In *Dirichlet orthogonality*, control functions are used to promote orthogonality next to the boundaries while the initial distribution of points along the boundary is kept constant. Zhang *et al.* (2008) explain that, compared to Dirichlet orthogonality, Dirichlet-Neumann orthogonality is more difficult to use especially for cases with complex boundaries. According to Kaul (2003), strictly boundary orthogonal grids, obtained from imposition of Dirichlet-Neumann orthogonality, can only be generated with partial control over the mesh spacing, while nearly-orthogonal grids allow for total control of the mesh spacing; these are obtained using Dirichlet orthogonality.

The iterative procedure necessary in elliptic grid generation methods requires an initial condition to start with. As described, algebraic generators are usually chosen for the initial grid, due to their low computational cost and ease of implementation. However, such methods may yield grids with overlap of lines of the same curvilinear coordinate. Such occurrence may not be problematic for elliptic generators of Laplace type, but may characterize a source of problems for an inhomogeneous generator since the range of initial conditions that would lead to convergence is substantially reduced in this case (Fletcher, 1996; Spekrijse, 1999). Thus, the method chosen for obtaining the initial grid is the parabolic grid generation technique proposed by Noack and Anderson (1990), due to its characteristic smoothness and low computational cost. On the other hand, an intrinsic characteristic of the initial parabolic grids is not allowing for control of coordinate spacing in all directions. As a consequence, the method proposed by Khaymaseh and Mastin (1996) for calculating the orthogonal functions during the iterative process had to be modified, once it keeps the line spacing of the initial grid. The full technique will be presented in a forthcoming article.

### 2.2 Relaxation Procedure, Implementation and Internal Functions

Computational cost for elliptic grid generation is a decisive issue that has been also reported by Zhang *et al.* (2008) and Kaul (2003). In the present work, in order to address computational cost issues, the following aspects have been emphasized: 1) The implementation has been performed in an object-oriented fashion using C++; 2) The Alternating Direction Implicit (ADI) method, proposed by Ballhaus *et al.* (1977a,b), has been used as relaxation method; and 3)  $\alpha$ -sequence, as described in Holst (1985), have been used in order to accelerate convergence. Moreover, the internal function calculation process is somewhat automatized in order to make it more usable and speed-up computational time.

A slightly modified version of the renowned Thompson's control functions is used (Thompson *et al.*, 1974). Furthermore, the method presented can be fully extended to 3-D, in which case the computational costs become even more important. About the use of object-oriented programming, the standard approach (which is well documented) has been used, and no further observations are necessary. With regard to ADI method, it has been implemented as proposed in the cited series of articles by Ballhaus *et al.*. Complementary information, such as a Von-Neumann stability analysis, a discussion regarding the use of the  $\alpha$ -sequence, as well as relevant usages may be found in Holst (1985) and Holst (1997). The internal function calculation process is addressed in the following section.

### 2.3 Internal Control Functions Calculation

As pointed by Khaymaseh *et al.* (1999), refined grids may be necessary to solve with precision the large gradients present in boundary layers and, although the elliptic generator system with orthogonality on boundaries provides for good characteristics in terms of smoothness and orthogonality, it may be still necessary to control the grid spacing. Thompson *et al.* (1977) proposed exponential control functions that have been reported to provide efficient control over the grid spacing. As argued by Thompson *et al.* (1985), the exponential control functions (the internal functions) may have to present absolute values significantly large to produce expressive effects on the grid. Larger amplitudes of the control functions may impact negatively on convergence. This is explained by Fletcher (1996), which also explains that large amplitudes may restrict possible initial conditions that lead to convergence. The internal control functions used here are a slightly modified version of those proposed by Thompson *et al.* (1977). The basic idea is to work around the problem at hand by "normalizing" the author's functions. To understand the methodology, consider the following inhomogeneous elliptic system which, similarly to Eq. (1), has also been widely used for grid generation (Spekreijse, 1999),

$$\begin{aligned}\nabla^2\xi &= \frac{a_{22}}{J^2}P(\xi, \eta); \\ \nabla^2\eta &= \frac{a_{11}}{J^2}P(\xi, \eta).\end{aligned}\quad (3)$$

According to Thompson *et al.* (1985), the control functions for this system are allowed to assume values orders of magnitude smaller than the ones in system Eq. (1) for similar effects. Comparing Eq. (1) with Eq. (3) makes it obvious that multiplying the functions  $P$  and  $Q$  in Eq. (1) respectively by  $a_{22}/J^2$  and  $a_{11}/J^2$  allows the coefficients inputed by the user (the decay parameters) to be the same as if the system Eq. (3) was being used. The only drawback is that the computational cost for calculating the internal control functions is slightly increased. Thus, the functions used here are of the form

$$\mathbf{Z}_I(\xi, \eta) = -\mathbf{A} \left\{ \sum_{i=1}^n \mathbf{a}_i \text{sgn}(\zeta - \zeta_i) e^{(-\mathbf{c}_i |\zeta - \zeta_i|)} + \sum_{j=1}^m \mathbf{b}_j \text{sgn}(\zeta - \zeta_j) e^{[-\mathbf{d}_j \sqrt{(\zeta - \zeta_j)^2 + (\eta - \eta_j)^2}]}\right\}, \quad (4)$$

where  $\mathbf{A} = (a_{22}/J^2, a_{11}/J^2)^t$ . The decay parameters inputed by the user are  $\mathbf{a}_i = (a_i^1, a_i^2)^t$ ,  $\mathbf{b}_j = (b_j^1, b_j^2)^t$ ,  $\mathbf{c}_i = (c_i^1, c_i^2)^t$  and  $\mathbf{d}_j = (d_j^1, d_j^2)^t$ .

$$\text{sgn}(x) = \begin{cases} 1, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ -1, & \text{if } x < 0. \end{cases} \quad (5)$$

It should be noted that the modification does not change the way  $\mathbf{Z}_I$  work and how they affect the system, which is well described in Thompson *et al.* (1977). Thompson *et al.* (1982) provide for a particularly good description of how to use the internal control functions to control the grid. It is worth mentioning that  $\mathbf{Z}_I$  from Eq. (3) are also compatible with the methodology proposed by Steger and Sorenson (1979).

The methodology proposed by Fletcher (1996) for imposing the internal control functions after the system has converged with  $\mathbf{Z}_I = 0$  has been automated as follows. The user is required to input the number of times the system will be iterated until it reaches convergence, called  $EI$  for *external iterations*. For each external iteration,  $\mathbf{Z}_I$  have its decay parameters  $\mathbf{a}_i$  and  $\mathbf{b}_j$  multiplied by  $(ne - 1)/EI$  ( $ne$  is the current external iteration number), and the system with the current  $\mathbf{Z}_I$  its iterated until convergence is obtained; the converged solution is used as initial condition for the next external iteration. The process is repeated from  $ne = 1$  until  $ne = EI + 1$ , in order that for the first external iteration, the system is iterated with  $\mathbf{Z}_I = 0$ , and for the last external iteration  $EI + 1$ , the system is iterated with the coefficients for  $\mathbf{Z}_I$  inputed by the user, necessary to obtain the final grouping of points and/or lines. Finally, the merge of internal and orthogonal control functions will be presented in a forthcoming article, with the full methodology.

## 3. RESULTS

The grids are evaluated based upon two standard academic criteria, namely, orthogonality and smoothness, represented respectively by the following indicators: deviation from orthogonality and grid aspect ratio, as proposed in Zhang *et al.*

(2008). Figure 1 presents the grid obtained with the present method for an academic geometry widely used in the literature (Khaymaseh *et al.*, 1999). The domain is defined by  $0 \leq x \leq 1$ , and  $0 \leq y \leq 0.75 + 0.25 \sin(\pi(0.5 + 2x))$ . For the present case, the grid size is  $51 \times 41$  cells, with a maximum aspect ratio of 4.5 and maximum deviation from orthogonality of 41.3 deg. As one should expect, deviation from orthogonality at the boundaries is minimum. It should be emphasized that, according to Zhang *et al.* (2008), elliptic grid generation systems, with Dirichlet boundary conditions, present a tendency of pronounced mesh squeezing at the concave boundary. However, as one can see in Fig. 1, the present approach seems to attenuate this effect, which is a very positive aspect of the formulation here implemented.

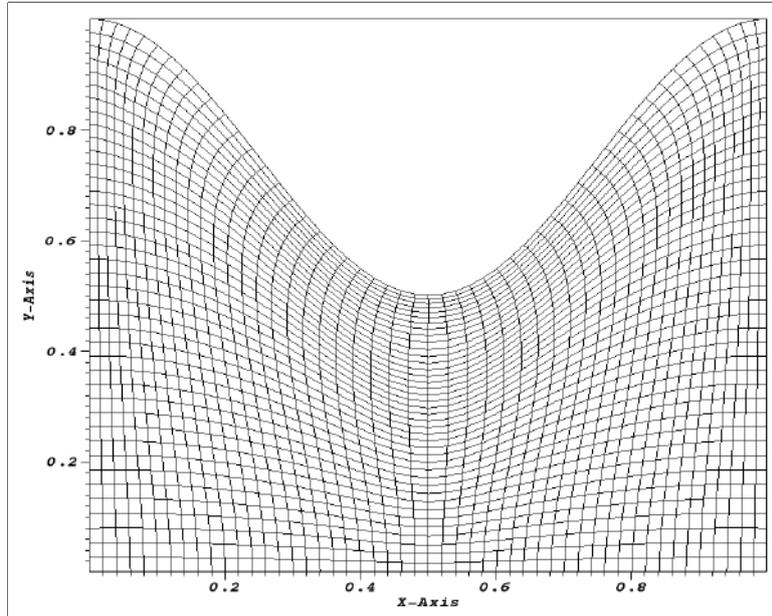


Figure 1. Grid for an academic geometry widely used in the literature.

Figures 2 and 3 show the grid for a domain bounded by three concave parabolic curves and one straight line, which corresponds to the region occupied by air in the space bounded by the fins and fins' base. The grid size in this case is  $151 \times 141$  cells. The mesh presents good properties, such as aspect ratio close to unity for most elements, with few elements distancing from such behavior. Maximum aspect ratio of 2.5 is reported in this case. Deviation from orthogonality follows the same tendency: few elements are not within 20% deviation. Moreover, as can be seen, the grid is nearly orthogonal to the boundaries with exception of corner regions.

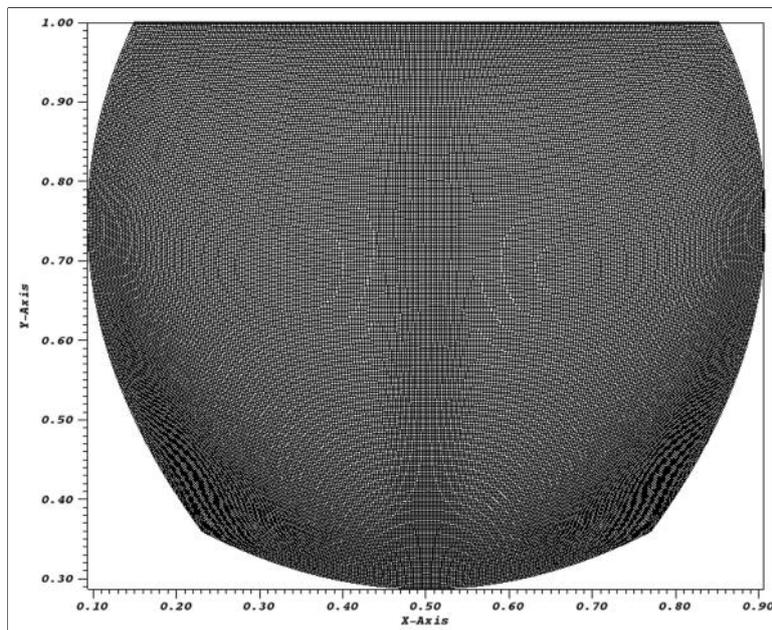


Figure 2. Grid for domain bounded by three parabolic curves and one straight line.

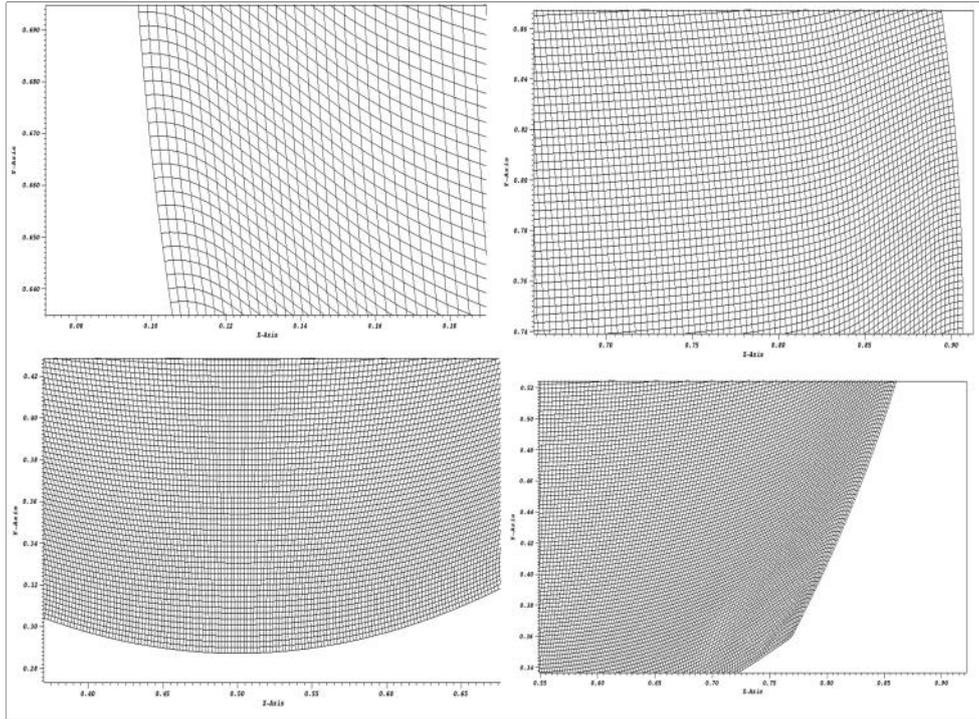


Figure 3. Details from grid of Fig. 2.

Figure 4 presents the grid for the complementary region of Fig. 2, this is, for the fins and their base. In the present case, the grid size is  $551 \times 55$  cells, with maximum aspect ratio of 4.5 and maximum deviation orthogonality of  $47.9^\circ$ . Once again, the grid orthogonality deviation on the boundaries is minimum, with exception of corner regions, and tendency of pronounced mesh squeezing at concave boundaries is again attenuated.

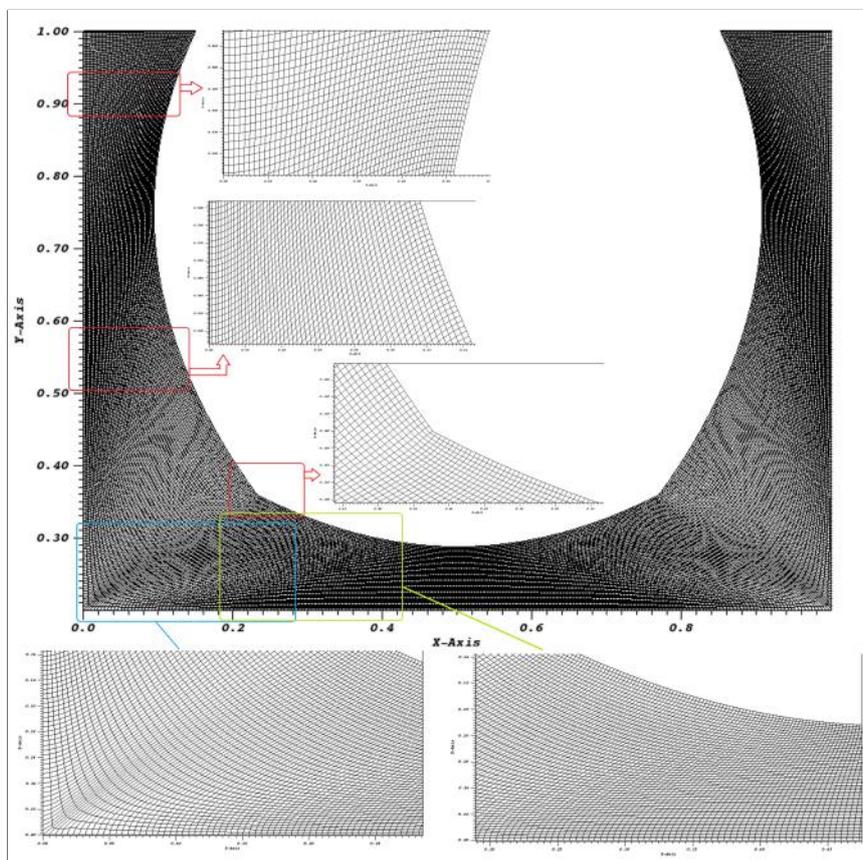


Figure 4. Elliptic grid for the fins.

#### 4. CONCLUDING REMARKS

In CFD analysis and mesh generation, complex geometries remain a challenge. In this study, that is the main concern. The mapping problem regarded as the most difficult in the literature, corresponding to the construction of an coordinate system fitted to a boundary of given shape, with a prescribed distribution of coordinate nodes along this boundary is addressed. The method proposed consists of an inhomogeneous elliptic system designed envisioning flexibility and reliability. The mapping technique proposed provide a method for the construction of a boundary-fitted coordinate system, nearly-orthogonal to the boundaries, in two dimensions; the method can be extended to 3D. The inhomogeneous terms are used in two ways: first, to obtain the near-orthogonality along the boundaries and two, to allow grid clustering inside the domain. In the last case, a simple modification of Thompson's control functions confirmed the expectations, simplifying the choice of the decay parameters by the user. As a drawback, the meshes obtained are not in general orthogonal over the entire domain, which may be desired in some applications. On the other hand, the flexibility obtained with the method is certainly an interesting asset, especially for complex geometries. Also, grids with Dirichlet boundary conditions are obtained, allowing the boundary point distribution to be controlled by the analyst. As commented, inhomogeneous elliptic generation systems are sensible to initial conditions, reason why a parabolic generation method has been chosen, which has proved to work fairly well. With regard to computational cost, the Alternating Direction Implicit scheme, with the appropriate  $\alpha$ -sequence, has shown to be an interesting choice, in special given the necessity of internal control functions for grid clustering, and mostly in case of a 3D application. Although not fully-automatic, the method has the advantage of allowing the analyst, who is cognizant with the specific application, to use its best judgment in order to obtain an optimized grid to the problem. The results have shown that boundary-nearly-orthogonal grids, with control over point distribution along boundaries, as well as control of point and line spacing inside the domain, can be obtained with the method proposed at a reasonable computational time.

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