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PERFORMANCE OF FINITE VOLUME DISCRETIZATION SCHEMES FOR THE CONVECTIVE-DIFFUSIVE LINEAR TRANSPORT EQUATION. PART II: HIGH EIGENVALUE-PECLET RATIOS

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Abstract. *The objective of this paper is to expand the investigation of numerical performance of various finite-volume discretization schemes, for the case of two-dimensional transport of an inert scalar in a constant velocity field. Central differencing, Simple Exponential, First Order Upwind, Second Order Upwind, QUICK, LOADS and UNIFAES schemes were submitted to a series of test cases given by distinct solutions of the transport equation to investigate the accuracy of the numerical schemes in a general situation. The governing equation of this problem has six elementary solutions in real form dependent on an eigenvalue. These exact solutions are imposed as Dirichlet condition at the boundaries nodes of the square domain. The system of equations generated by each discretization scheme is solved employing the ADI method. QUICK, UNIFAES and LOADS schemes present the best performance in most cases, however, QUICK may present non-monotonic convergence. Generally the numerical error of all schemes grows with increasing eigenvalues and tends to become constant at high frequencies. However, regions of the spectrum of solutions with low numerical errors are formed for some schemes.*

Keywords: *Finite Volumes, Discretization Schemes, Numerical Analysis, Eigenvalues, CFD.*

1. INTRODUCTION

Convection-diffusion processes are inerent to viscous flow and correlated transport phenomena, so arising frequently in many areas of applied sciences and engineering. In the discretization of convection-diffusion transport equations, the three classical choices for the numerical solution of partial differential equations (PDE's) are the finite difference method (FDM), the finite element method (FEM) and the finite volume method (FVM). Although Finite difference and Finite element methods have been employed in discretizing convective-diffusive equations, the finite volume approach has been overcoming the other methods in solving fluid flow problems. The reasons are the direct physical appeal of the method, its relative algebraic simplicity and particularly the conservative form of the method which is capable of conserving the fluxes independent of mech refinement.

The choice of a discretization scheme has been a dominant subject in the literature when it comes to the numerical solution of problems involving transport phenomena and fluid mechanics. The most used first order scheme is the First Order Upwind (FOU) proposed by Courant and Rees (1952). The FOU scheme it's simple, stable and provides a smooth solution, however it is excessively diffusive for problems of moderate to high Peclet numbers, what results in inaccurate solutions.

The Central Differencing is the simplest second order scheme, but presents numerical instability and spatially oscillatory solutions as the Peclet number increases. To overcome these limitations new second order schemes have been developed, as the Second Order Upwind scheme (SOU) presented by Shyy (1985) and the quadratic upstream interpolation for convective kinematics (QUICK) developed by Leonard (1979). These methods do not present the stability problems like the central scheme, however in some cases QUICK exhibits non-monotonic convergence.

Among these options there are also the exponential-type schemes, where the interpolation functions are obtained from the exact solution of a linear equation which approximates the equation of interest. The exponential-type schemes are so called because the exponential function always appears in their interpolating curves. All exponential-type schemes are asymptotically second-order. The first FVM exponential-type scheme is the Simple Exponential of Spalding (1972) and Raithby and Torrance (1974), the most sophisticated schemes of this class are the Locally Analytic Differencing Scheme (LOADS) of Wong and Raithby (1979) and the Unified Finite Approaches Exponential-type Scheme (UNIFAES) proposed by Figueiredo (1997).

This paper together with is companion paper of Nascimento *et al.* (2018) compose an investigation of the performance of referred numerical methods in the case of the two dimensional transport of a inert scalar in a parallel uniform velocity flow. This test problem is governed by a linear equation, solved by the method of separation of variables, that has six types of elementary solutions in real form dependent on an eigenvalue (λ). The previous paper is focused on analysis of low λ/Pe ratios for each solution and also investigate the effects of the flow angle and the Peclet number variation on the numerical accuracy of the discretization schemes. The present paper extends that analysis to general λ/Pe ratios, through variations of the eigenvalue of each solution and also presents how this variation affect the numerical accuracy of this schemes.

2. LINEAR TEST PROBLEM

For simplicity reasons, the schemes will be derived with the two-dimensional transport equations with constant properties in non-dimensional form as shown in Eq. (1), where Pe is the global Peclet number and t represents the non-dimensional time. This is the simplest two-dimensional transport equation, and has been used in test cases by many authors in recent years.

$$\frac{\partial \phi}{\partial t} + \frac{\partial (Peu\phi)}{\partial x} + \frac{\partial (Pev\phi)}{\partial y} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{1}$$

Integrating Eq. (1) on the cell area on the right of Fig. 1 and employing the divergence theorem according to the FVM methodology the discretized equation of this problem, given by Eq. (2), is obtained. Where J_w is the sum of the convective and diffusive fluxes at the interface w, given by Eq. (3), and so on for the other cell boundaries.

$$\frac{\partial \phi}{\partial t}\Delta x \Delta y + (J_e - J_w)\,\Delta y + (J_n - J_s)\,\Delta x = 0 \tag{2}$$

$$J_w = Peu_w\phi_w - \left.\frac{\partial \phi}{\partial x}\right|_w \tag{3}$$

The discretizing schemes will evaluate these fluxes in different ways, details and the algebraic procedures of the discretizing schemes SOU, QUICK, LOADS and UNIFAES can be found in Nascimento *et al.* (2018).

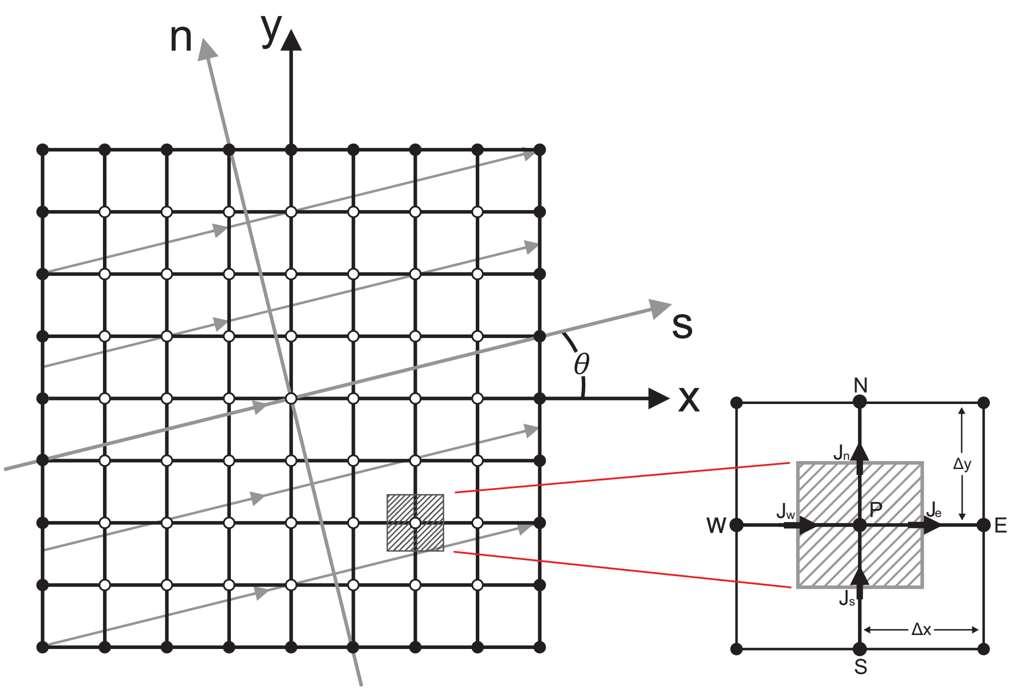


Figure 1. Exact and numerical coordinates system.

In order to obtain the exact solution of Eq. (1) it is simpler to perform a coordinate change of the system (x, y) for the system (s, n), parallel and normal to the direction of the flow as can be observed in Fig. 1. The system of coordinates (s, n) is adopted for analytical solutions, while the system (x, y) represents the numerical axis, which may or may not be aligned to the flow and that obeys the Eq. (4).

$$Pe\frac{\partial \phi}{\partial s} - \frac{\partial^2 \phi}{\partial s^2} - \frac{\partial^2 \phi}{\partial n^2} = 0 \tag{4}$$

The schemes mentioned in previously section were submitted to a series of steady-state linear test cases given by distinct solutions of the transport equation on a constant velocity field defined by modulus V and angle θ with the x-axis, so that its components are $u = V.cos(\theta)$ and $v = V.sin(\theta)$.

The Eq. (4) is solved analytically by the method of separation of variables, details can be obtained in companion paper of Nascimento *et al.* (2018) and in Figueiredo (1997). The elementary solutions of Eq. (4) are presented below, all elementary solutions depend on an eigenvalue (λ). The solutions given by Eq. (5) and (6) are valid for any ratio of λ/Pe.

$$\phi_A = exp\left(\frac{Pe - \sqrt{Pe^2 + 4\lambda^2}}{2} s\right) sin(\lambda n) \tag{5}$$

$$\phi_B = exp\left(\frac{Pe + \sqrt{Pe^2 + 4\lambda^2}}{2} s\right) sin(\lambda n) \tag{6}$$

The solutions given by Eq. (7) and (8) are valid only when $\lambda/Pe \leq 0.5$.

$$\phi_C = exp\left(\frac{Pe - \sqrt{Pe^2 - 4\lambda^2}}{2} s\right) exp(\lambda n) \tag{7}$$

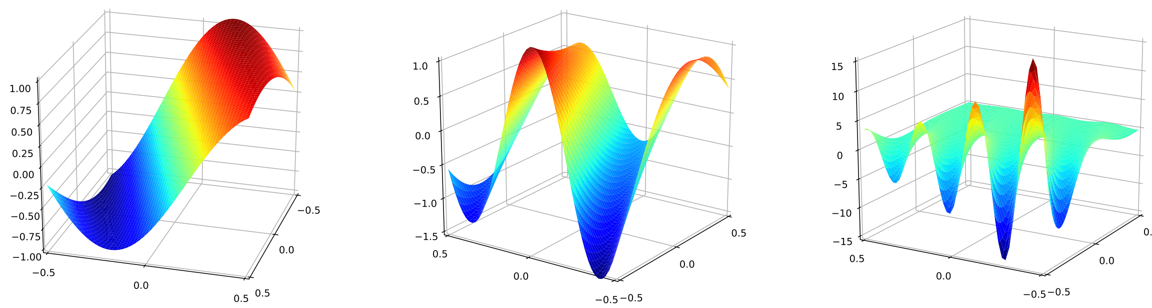
$$\phi_D = exp\left(\frac{Pe + \sqrt{Pe^2 - 4\lambda^2}}{2} s\right) exp(\lambda n) \tag{8}$$

For relations above this limit, in $\lambda/Pe > 0.5$, the discriminant of the square root becomes negative, leading to a complex exponential, the real form solution to these cases are given by Eq. (9) and Eq. (10).

$$\phi_{CD} = exp\left(\frac{Pe}{2} s\right) sin\left(\sqrt{\lambda^2 - \left(\frac{Pe}{2}\right)^2} s\right) exp(\lambda n) \tag{9}$$

$$\phi_{DC} = exp\left(\frac{Pe}{2} s\right) cos\left(\sqrt{\lambda^2 - \left(\frac{Pe}{2}\right)^2} s\right) exp(\lambda n) \tag{10}$$

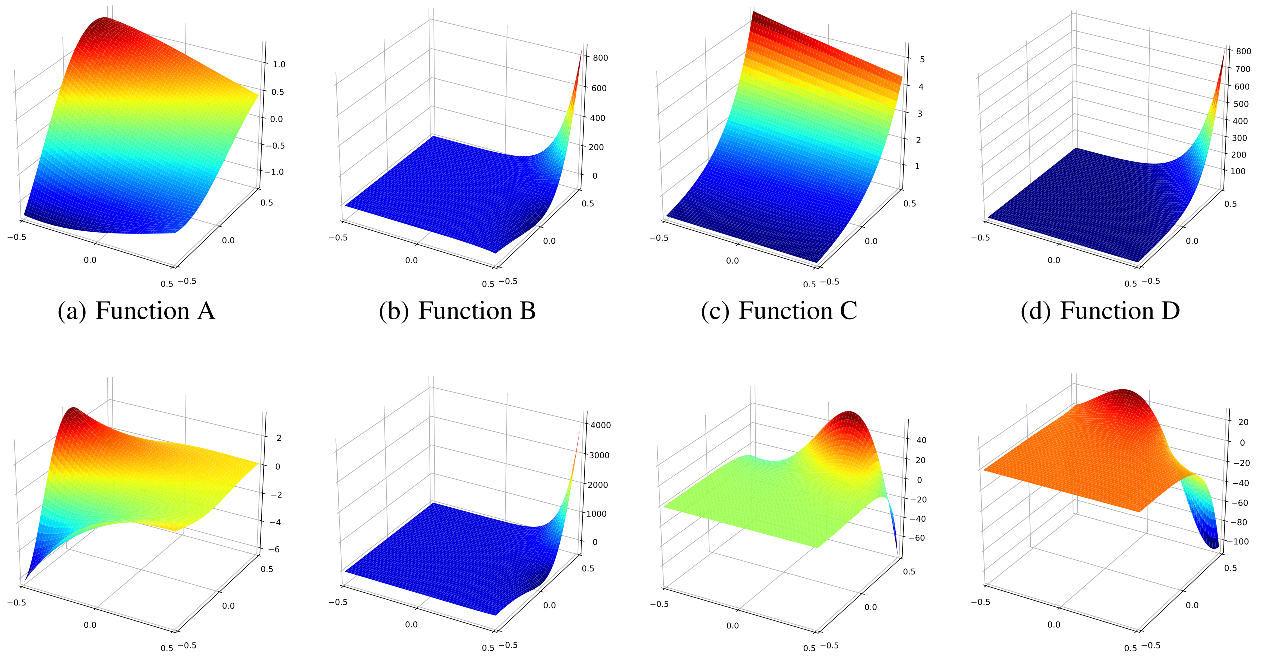
All six elementary solutions are dependent on an eigenvalue, as mentioned above, Fig. 2 shows the influence of the eigenvalues on the solution of type A as the frequency increases, where it can be observed that the solution of type A is very sensitive to the variation of frequency. Later in this paper will be investigate, in the six types of solutions, if this influence of the eigenvalue variation affects the accuracy of the numerical schemes tested in the present paper.



(a) $\lambda = \pi\sqrt{2}$ (b) $\lambda = 3\pi\sqrt{2}$ (c) $5\pi\sqrt{2}$

Figure 2. Metamorphosis of solution type A by varying the eigenvalue at $Pe = 100$ and $\theta = 22.5$

Figure 3 shows the six elementary solutions presented below on two cases, at λ/Pe ratios smaller and greater than 0.5 respecting the domain where each solution is valid. Thus solutions of type C and D are presented only for ratios lower than 0.5 and solutions CD and DC are presented only for ratios above this limit. Figures 3(a) to 3(d) shows the solutions of types A, B, C and D at $Pe = 10, \lambda = 3$. Figures 3(e) to 3(h) shows the solutions of types A, B, CD and DC at $Pe = 10, \lambda = 6$.



(a) Function A (b) Function B (c) Function C (d) Function D

(e) Function A (f) Function B (g) Function CD (h) Function DC

Figure 3. Elementary solutions of transport equation at $Pe = 10$ and $\theta = 22.5$.

The Solutions of type A and B are sinusoidal in the normal direction to the flow and exponentials in the direction of flow, and they differ from each other in relation to the smooth or abrupt nature of the exponential function. Analogous situation occurs between types C and D, which are both exponentials in cross-flow direction. The functions A and C tend to represent diffusion predominantly crossed to the flow, and types B and D to simulate cases with diffusion in the counter current direction, particularly intense in a boundary layer in the exit. The CD and DC types characterize situations where diffusion occurs both counter current and in the normal direction of the flow. A companion paper of Nascimento *et al.* (2018) considered low $\lambda/Pe(\leq 0.1)$ ratios, so being restricted to functions A,B,C and D. The present paper expands this analysis to higher λ/Pe ratios, so including also functions CD and DC.

3. RESULTS AND DISCUSSION

The exact elementary solutions of the convective-diffusive transport equation,Eq. (5) to (10), are used to perform a series of tests of the seven finite volume schemes mentioned earlier, which are central Differencing, Simple Exponential, First Order Upwind, Second Order Upwind, QUICK, LOADS and UNIFAES.

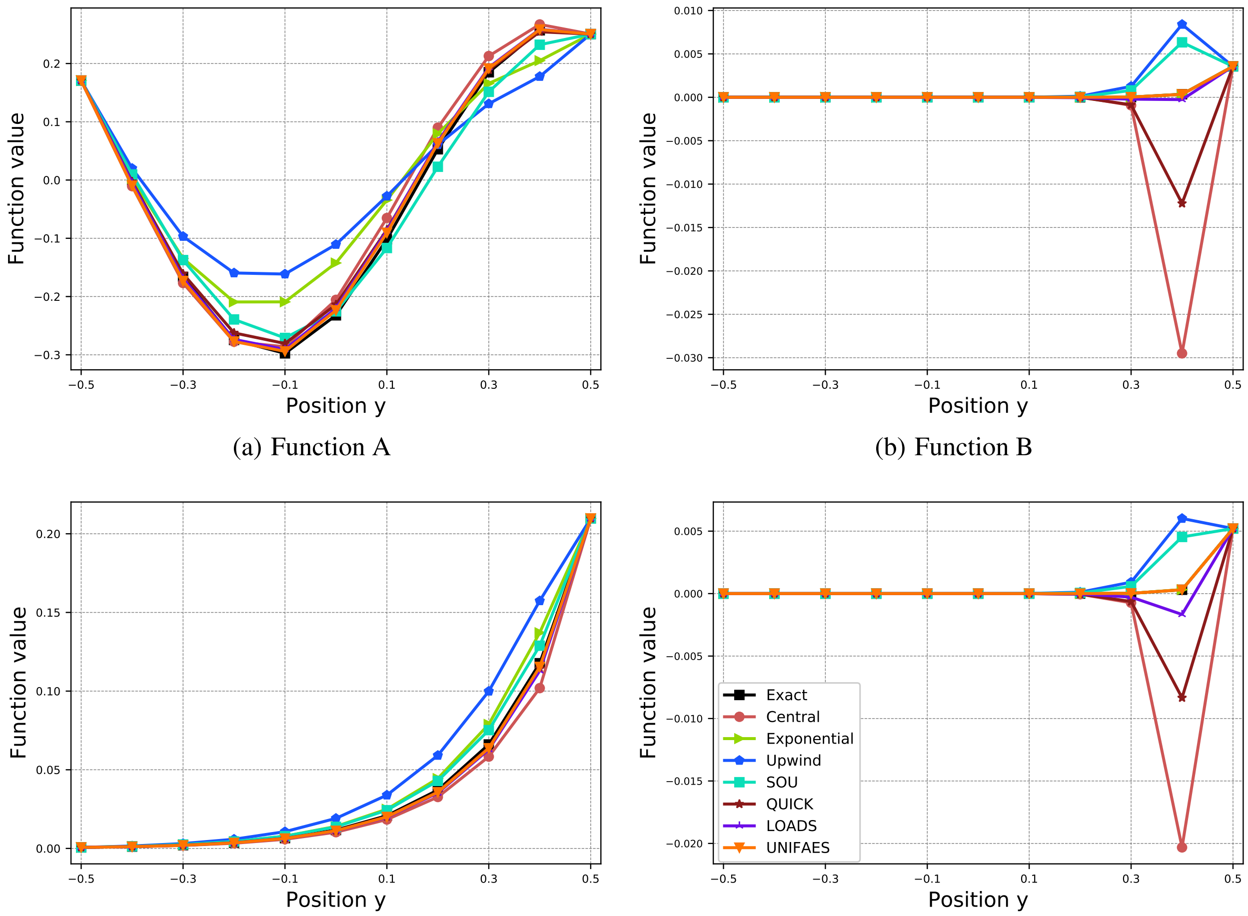
The exact solutions are imposed as a Dirichlet condition at the boundaries nodes of the square domain with square cells, shown in Fig. 1. The system of equations generated by the discretization of Eq. (1) is solved employing the transient Alternating Direction Implicit (ADI) method for the five-point schemes. The SOU and QUICK schemes are computed by upwind-based ADI and the extra terms are treated explicitly while the UNIFAES and LOADS are computed as the Simple Exponential with the extra terms computed explicitly. The mathematical formulation of the central scheme was used for the QUICK and SOU schemes at the entrance boundaries of the domain where terms related to nodal points would be outside the domain. The angle between the grid and flow direction is set at 22.5 degrees, avoiding angles such as 0 or 45 degrees whose symmetry can lead to particular, unwanted situations. The influence of the angle variation is depicted in the companion paper of Nascimento *et al.* (2018).

The time is used to control the iterative process, only the steady state solution is obtained. The stopping condition adopted was the maximum rate of error variation of the function with time being less than 10^{-5}. The errors are evaluated by the root mean square (RMS) error and normalized by the maximum distance between the values of the solution within the domain, the same normalization was used in all analyzes presented below.

3.1 Behavior of schemes at low and high eigenvalue ratios

This first approach investigates the behavior of numerical schemes for ratios (λ/Pe) of 0.1 and 0.6 respectively. Figure 4 shows the profiles of the exact and the numerical schemes solutions for existing functions A, B, C and D at $Pe = 60$ and $\lambda = 6$ in 10x10 spacings grid. The CD and DC functions are not presented for these parameters, since the $\lambda/Pe(= 0.1)$ ratio is less than 0.5 and these solutions are valid for ratios greater than 0.5, as informed before. All comparative figures of the present analysis, shows the column or row of the solution where the largest error are usually found next to the exit boundary.

The discretization schemes analyzed behave in a similar way for the solutions of functions B, C and D. Where UNIFAES, QUICK, LOADS, SOU and Simple Exponential perform well for this functions under this conditions. These schemes behaved as stable and not very diffusive in these cases. Unlike upwind and central, as expected they did not present good approximations in relation to the exact solution. The central scheme in all solutions behaved as highly oscillating and not precise. For function type A, shown in Fig. 4(a), most schemes were diffusive except for UNIFAES, QUICK and LOADS that practically coincided with the exact solution. It should be noted that LOADS and UNIFAES, both schemes of the exponential-type family, as well as QUICK present small errors in practically all functions of this case.



(a) Function A

(b) Function B

(c) Function C

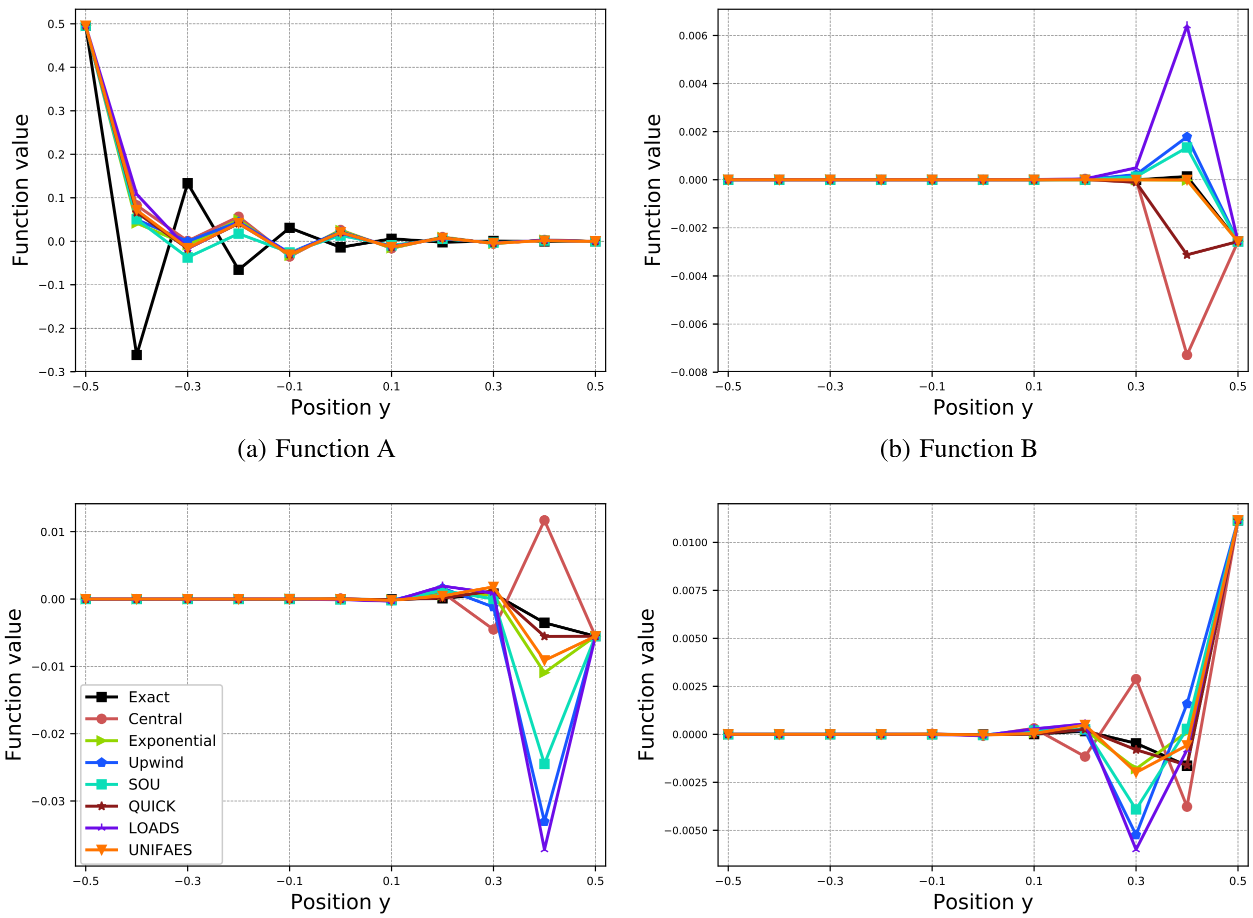
(d) Function D

Figure 4. Exact and numerical profiles for functions A,B,C and D at 10x10 grid, $x = 0.4$, $Pe = 60, \lambda = 24, \theta = 22.5°$.

It can be observed that the functions A and C were the ones in which the schemes presented the biggest errors in relation to this exact solutions. In general, QUICK, LOADS and UNIFAES schemes performed better than the five-node schemes. For these conditions in function A the errors are 0.91%, 0.62% and 0.48% for QUICK, LOADS and UNIFAES respectively, UNIFAES is the most precise scheme in this condition. On the other hand, in function C, QUICK was the best scheme along with UNIFAES and LOADS with respectively 0.101%, 0.108% and 0.209% of error. It can be observed that, for this four types of solutions, UNIFAES is always among the schemes that present the best results, when it is not the best.

Figure 5 presents the profiles of the exact and the numerical schemes solutions for functions A, B, CD and DC with $Pe = 60$ and $\lambda = 36$ in 10x10 spacings grid. The C and D functions are not presented for these parameters, since the

$\lambda/Pe(=0.6)$ ratio is greater than 0.5 and these solutions are not valid for such ratios. The tested schemes behaved as stable and not very diffusive in functions B, CD and DC generally presenting good approximations, in this first analysis, probably due to the low Peclet number adopted in these situations to avoid problems with high frequencies (λ) which will be portrayed later. On the other hand, for function A all discretization schemes show diffusive behavior in relation to the exact solution, evidencing the first clue that the numerical accuracy of the schemes is affected with the increase of the eigenvalue.



(a) Function A (b) Function B (c) Function CD (d) Function DC

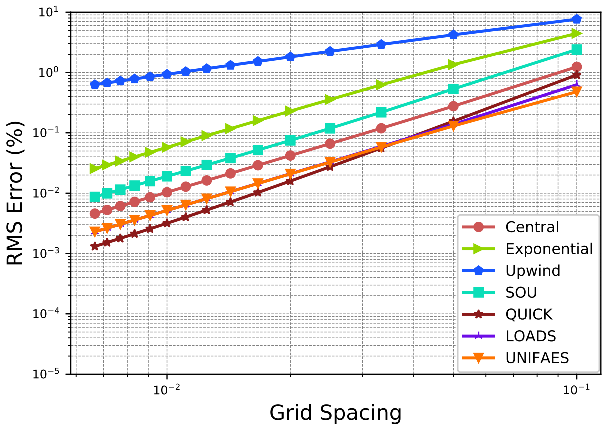
Figure 5. Exact and numerical profiles for functions A,B,CD and DC at 10x10 grid, $x = 0.4$, $Pe = 60, \lambda = 36, \theta = 22.5°$.

In functions B, CD and DC practically all the schemes presented an slight oscillatory behavior at the end of the curve, this was already expected for central, FOU and SOU scheme. However LOADS emerge as the most oscillating scheme in this case, even more than the central scheme, also featuring such demeanor at the end of the curve. In his investigation, Prakash (1984) also found some cases where LOADS presented wiggly behavior. The performance of UNIFAES it's very close to Simple Exponential in functions A, B CD and DC, without abrupt fluctuations.

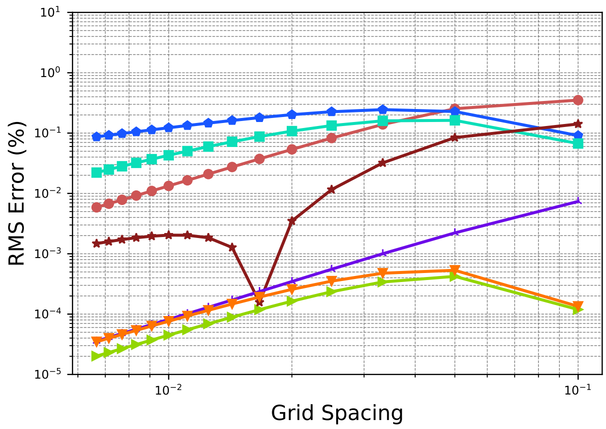
For this case the functions A and CD were the ones in which the schemes presented the biggest errors in relation to this exact solutions. For these conditions in function A the errors are 4.61%, 5.52%, 4.84% and 4.15% for the QUICK, LOADS, UNIFAES and Simple Exponential respectively, the last one is the most precise scheme in this condition. In the CD function the QUICK scheme is the most precise scheme, followed by the UNIFAES and the Simple Exponential, with respectively 0.02%, 0.06% and 0.08% of error. The same behavior of the schemes in CD function occurs in DC where again the QUICK scheme is the most precise scheme, followed by the UNIFAES and the Simple Exponential. The central scheme did very well here due to the low Peclet number adopted, because of the limitation of assuming high frequencies, on the relation of λ/Pe, in these solutions. In the solutions presented in Fig. 4 and Fig. 5 the UNIFAES is always among the schemes that present the best results. It can be observed that, for low and high ratios of λ/Pe, UNIFAES and QUICK comport very well in these situations, without abrupt fluctuations or differences of the exact solution. To obtain better conclusions about the behavior of these schemes under these conditions, one should analyze the error of the schemes according to the refinement of the grid.

Figure 6 show the results of the RMS error by its respective level in the grid for each scheme and the functions A, B, C and D at $Pe = 60, \lambda = 24, \theta = 22.5°$. The errors are expressed as a percentage of the difference between exact and computed values. In function A all differencing schemes, with the exception of upwind, show effectively quadratic reduction of the error with the grid refinement. While the Simple Exponential, SOU and the central scheme shows higher numerical errors, the superiority of UNIFAES, LOADS and QUICK is maintained with refinement for this function showing the lowest errors tending asymptotically to the quadratic behavior with the grid refinement.

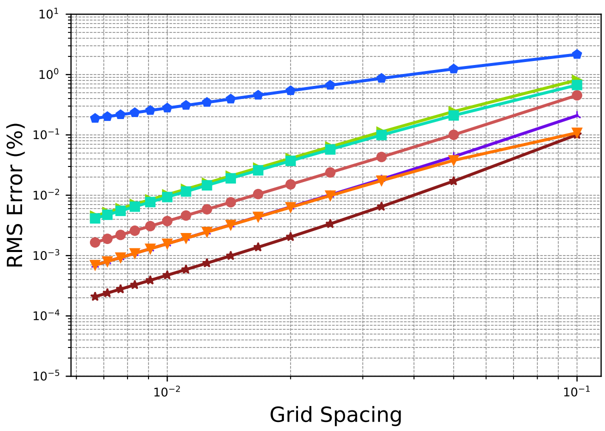
For functions B, unlike the function A, the asymptotic tendencies can be noticed only at very refined grids. The Simple Exponential, SOU, LOADS and UNIFAES behave more closely to the quadratic, while QUICK have his behavior dominated by non quadratic convergence. The asymptotic errors of UNIFAES and LOADS appear to coincide or to be very close between themselves at refined grids. The behavior of the schemes in functions C and D is analogous to functions A and B respectively.



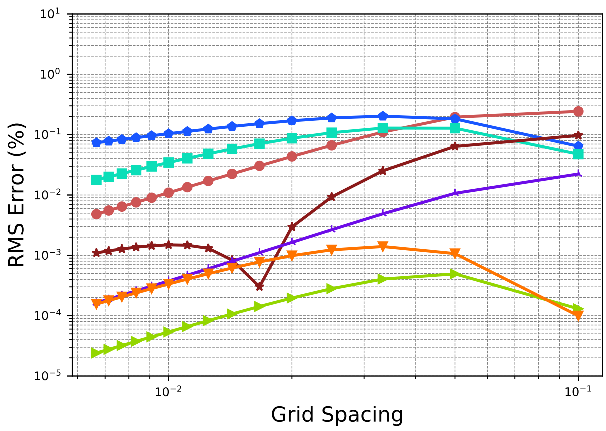
(a) Function A



(b) Function B



(c) Function C



(d) Function D

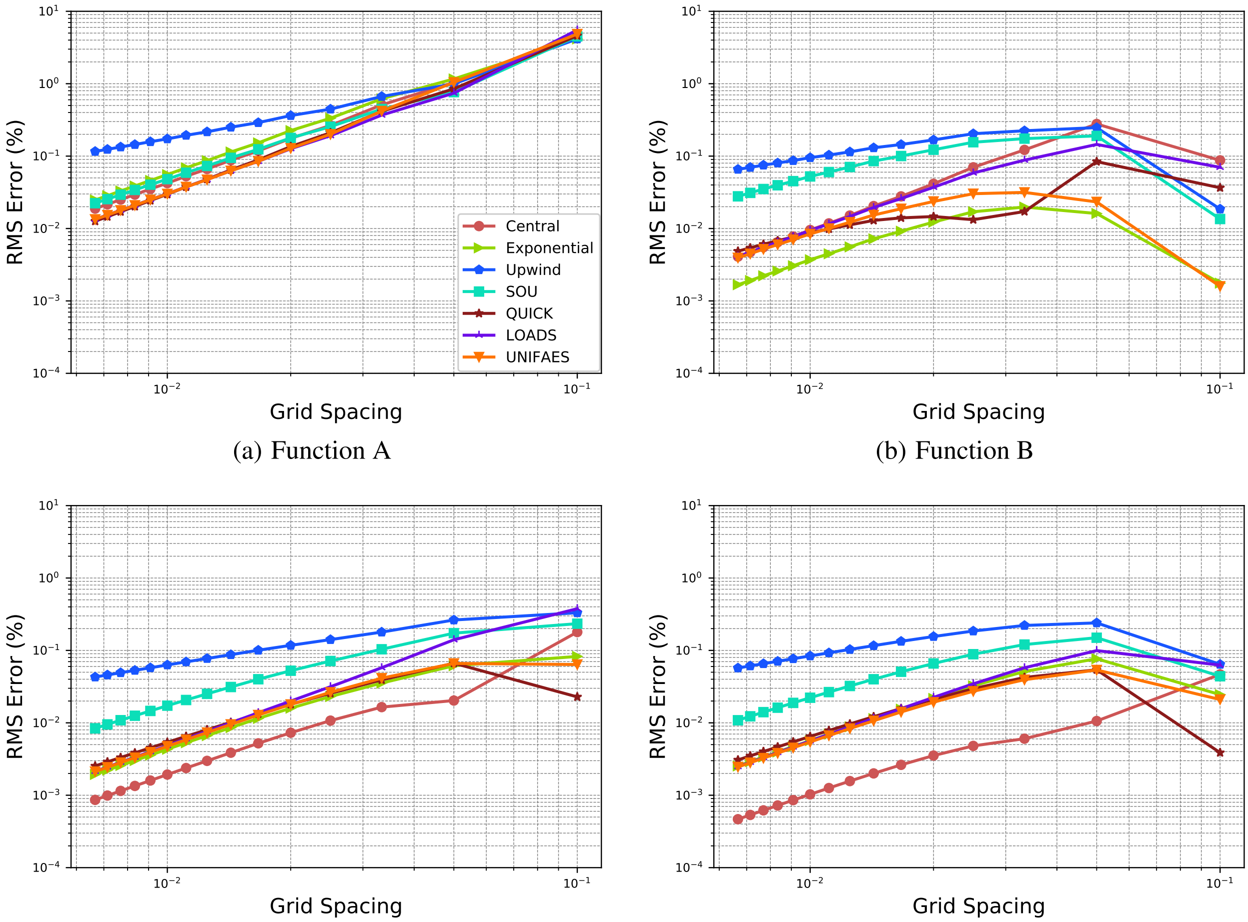
Figure 6. RMS Error Dependence on grid refinement for functions A,B,C and D at $Pe = 60, \lambda = 24, \theta = 22.5°$.

It can be observed that UNIFAES, LOADS and QUICK, undoubtedly present the best results in general, except for some situations where the Simple Exponential scheme did better, as can be seen in Fig. 6(b) and Fig. 6(d). The asymptotic errors of UNIFAES and LOADS practically coincide for all functions tested for these parameters.

As expected, upwind presented the largest numerical errors in the four types of functions, due to its first-order accuracy. The quadratic behavior of the second order schemes was achieved on average for a $40x40$ grid. In functions B and D the QUICK scheme presented a non-monotonic convergence, it should be noted that the QUICK was accurate second-order in all results of this study and not third-order as proposed by Leonard (1979) this is shown by Taylor series analysis in Nascimento *et al.* (2018).

Figure 7 show the results of the RMS error against refinement level for each scheme in the functions A, B, CD and DC at $Pe = 60, \lambda = 36, \theta = 22.5°$. The behavior of the schemes in functions A and B for $\lambda = 36$ is analogous to $\lambda = 24$, in function CD all differencing schemes, with the exception of upwind, show effectively quadratic reduction of the error with the grid refinement. For function DC, like the function CD, almost all differencing schemes show quadratic reduction of the error with grid refinement, the asymptotic errors of UNIFAES, QUICK and LOADS appear to coincide or to be very close between themselves.

In both CD and DC functions the central differencing scheme is the most accurate scheme favored by the low Peclet number adopted. Again, QUICK, LOADS, UNIFAES and Simple Exponential have results with close accuracy. The SOU scheme shows its best performance in thick grids, when the refinement of the grid is moderate to fine its performance is better only than upwind. Upwind, as expected, presented first-order numerical error. The quadratic behavior of refinement of the second order schemes was achieved on average for a 30x30 grid, it should be emphasize once again that QUICK was accurate second-order, contradicting its proposal of being a third-order scheme.



(a) Function A

(b) Function B

(c) Function CD

(d) Function DC

Figure 7. RMS Error Dependence on grid refinement for functions A,B,CD and DC at $Pe = 60, \lambda = 36, \theta = 22.5°$.

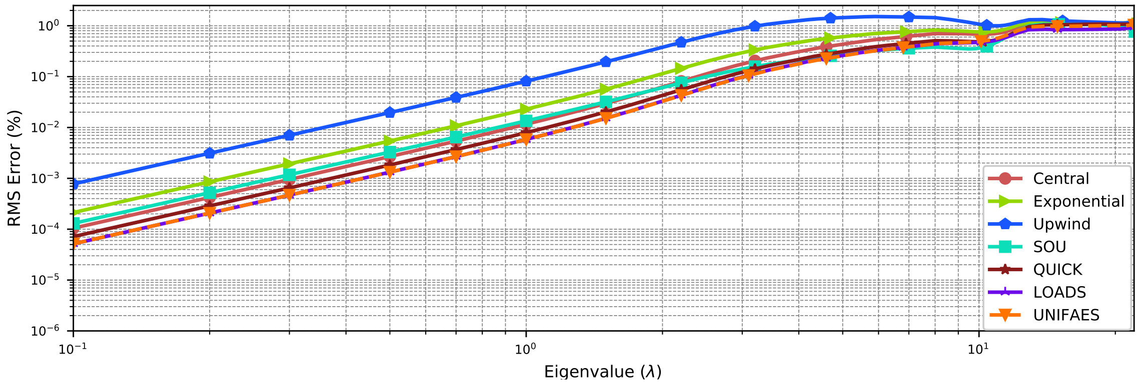
It should be observed that in the first case analyzed for $Pe = 60$ and $\lambda = 6$ where $\lambda/Pe = 0.1$ the obtained results are similar to those of Nascimento *et al.* (2018) for $Pe = 100$ and $\lambda = 10$, evidencing that the performance of the schemes depends not only on eigenvalue but on the relative position of the ratio λ/Pe in its spectrum as observed by Figueiredo (1997).

3.2 Investigation of schemes errors at growing eigenvalues

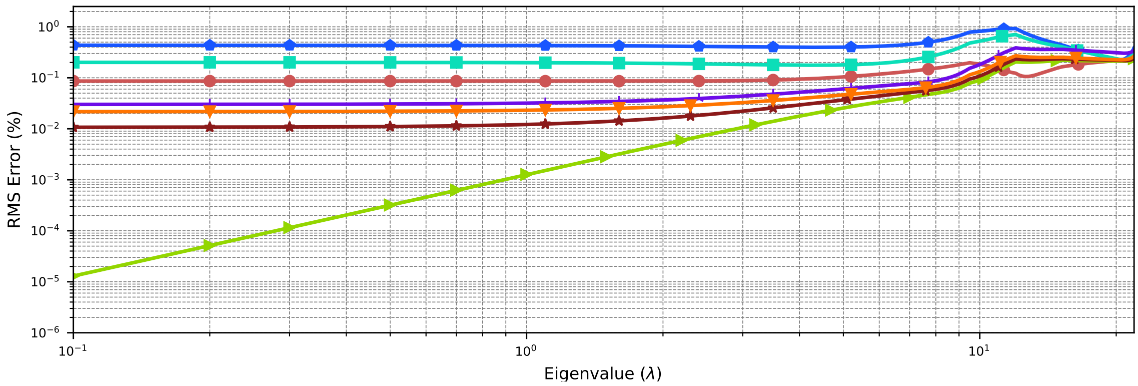
The investigation of the behavior of the schemes for growing eigenvalues is a challenging task, due the difficulty given by the inadequacy of the numerical grid to resolve high frequencies. The first approach followed here in order to avoid such difficulty is lowering the value of the Peclet number, lets say to $Pe = 10$. This way one can compare the behavior of the different schemes within a wider range of the ratio λ/Pe, keeping acceptable solutions. For such low Peclet number the errors are generally small, thus the unique available information about the numerical evaluation of the schemes it is their relative positions. Figure 8 present the results of the RMS error as function of the eigenvalue for each scheme and type of function in a 10x10 grid.

Figure 8(a) and 8(b) shows the errors for type A and B solutions in the frequency range between 0.1 and 20. It can be observed that in function A the errors of these schemes increase with increasing of λ/Pe and seems to approximate each other as the ratio of λ/Pe becomes greater than unity. In function B the errors are practically invariant with λ/Pe except for the Simple Exponential.

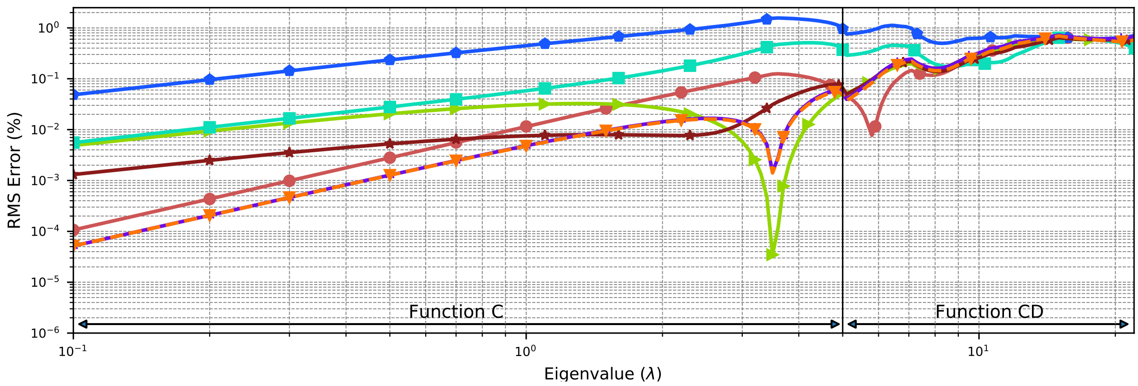
It should be pointed out that the schemes are more sensitive to the increase in eigenvalue in function A than in function B. In function B, with the exception of the Simple Exponential, the schemes will present changes in their numerical accuracy only at high frequencies, unlike function A where all errors of the schemes are equally affected by the increase



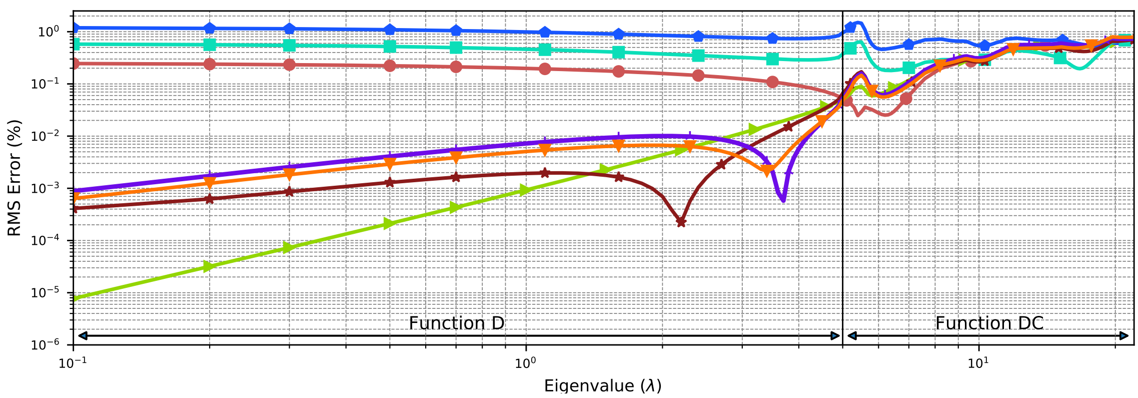
(a) Function A



(b) Function B



(c) Functions C and CD



(d) Functions D and DC

Figure 8. Schemes errors with growing eigenvalues $Pe = 10$, 10x10 spacings grid.

of the eigenvalue. It has been observed previously that the increase of the eigenvalue increases the numerical errors of the schemes, but in addition it is noticed that for the function A and B the relative performance of the schemes is maintained with the increase of the eigenvalue.

Figures 8(c) and 8(d) show the errors for type C, D, CD and DC solutions in a overall frequency range between 0.1 and 20. It should be noted that these figures are divided into two parts, the first part presents the numerical errors for the functions C and D in the frequency range of 0.1 to 5 which is the validity interval of the solution for the Peclet number adopted in this case. The second part shows the errors for the functions CD and DC in the frequency range between 5 and 20, respecting the validity interval of each solution, where there appear to be continuity between the C and CD functions as well between D and DC functions.

In Fig 8(c) and 8(d) the same observation that the error increases with the increase of λ/Pe can be made in this case. However in Function C, UNIFAES, LOADS and Simple Exponential schemes present regions of the spectrum of λ/Pe where the errors decays. The same thing can be observed in Function CD. Function D also have regions of the spectrum of λ/Pe where the schemes errors decrease, here the schemes that exhibit this are UNIFAES, LOADS and QUICK. The same occurs on function DC with a slight oscillation behavior. The central differencing followed by QUICK, SOU and UNIFAES are the schemes that present the smallest RMS errors for functions CD and DC. The Central Differencing scheme presented extremely satisfactory results in these functions, differently as in the functions A, B, C and D.

With few exceptions, there is a general increase in the errors of all schemes as the ratio λ/Pe increases, maintaining their relative positions but decreasing the relative distance between them. As seen in Fig. 8 all the seven schemes present roughly the same performance when this ratio (λ/Pe) is about or above unity. UNIFAES, LOADS, QUICK in average presents the smallest errors within a wide range of functions and ratios for all solutions.

In all cases where high frequencies are allowed, as in the functions A, B, CD and DC the errors of the schemes tend to no longer change with the increase of the eigenvalue. Regions of low numerical errors are formed for some schemes in the C, D, CD and DC functions. Now it is possible to observe in Fig. 8(c) and Fig. 8(d) why the central scheme presented such satisfactory results presents in Fig. 7(c) and Fig. 7(d) for functions CD and DC. This occurs because the central scheme it was situated in a region where the λ/Pe ratio for the analyzed parameters favors the scheme, in this region the central presents numerical errors lower than others discretizing schemes.

3.3 Investigation of schemes errors at high eigenvalues and high Peclet numbers

Another approach to investigate high frequencies without lowering the Peclet number is employing a purely convective transport of a step profile, obtained for the Fourier series of the square wave. This is a slowly converging test, where high frequency terms have strong influence. The convection of a stepwise profile is a severe test function for convection schemes, due to the high Peclet number and to the function discontinuity voids any conclusion based on Taylor series analysis.

The behavior of the schemes at high eigenvalues with high Peclet numbers can be investigated by employing a summation of functions A on the form given by Eq. (11), such configuration was also used by Figueiredo (1997). That produces the purely convective transport of a stepwise profile as both Peclet number and the number of terms in the series, M, tend to infinite.

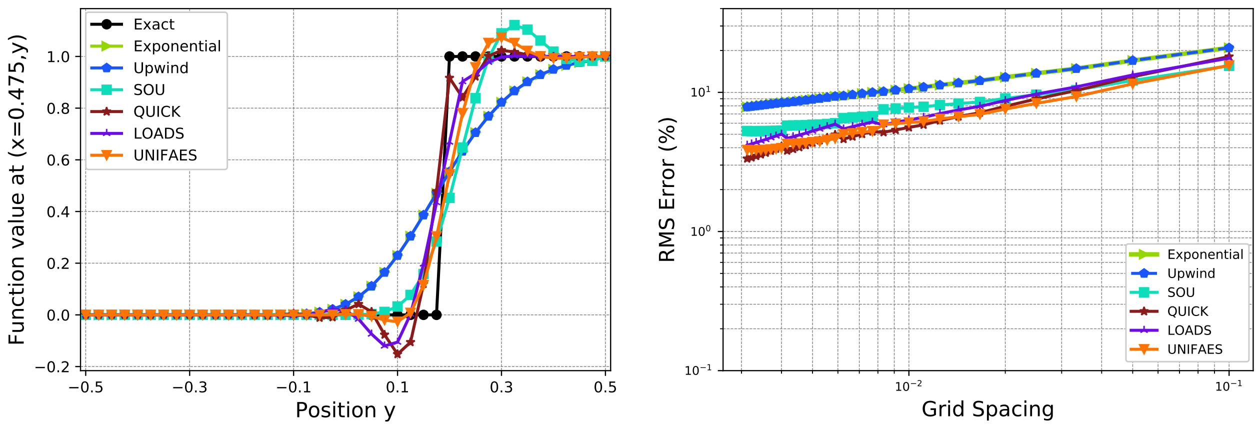
$$\phi = \sum_{m=0}^{M} exp\left(\frac{Pe - \sqrt{Pe^2 + 4\lambda_m^2}}{2}s\right)\frac{sin(\lambda_m n)}{(2m-1)} \tag{11}$$

where

$$\lambda_m = (2m-1)\lambda_0 \tag{12}$$

Figure 9(a) shows the performance of the six schemes presented in this paper for $Pe = 10^9, \theta = 22.5°$, $M = 500$, $\lambda_0 = \pi\sqrt{2}/2$ and a 40x40 spacings grid. The coinciding exponential and upwind schemes show diffusive behavior. All other solutions are closer to the exact, but present some overshoots or undershoots. LOADS and QUICK shows undershoots in the lower level region, with 12.0% and 15.2% of maximum error respectively. The SOU scheme presents 12.1% of maximum error in his overshoots and UNIFAES presents both undershoots and overshoots with respectively 2.6% and 7.3% of error for this profile, therefore its errors are smaller in the minimax sense. Furthermore, QUICK presents an oscillation in the result on the rise of the discontinuity.

Figure 9(b) shows the RMS error by its respective level in the grid for each scheme for the stepwise profile at $Pe = 10^9$ and $\theta = 22.5°$. It is possible to observe a non-monotonic behavior in all schemes from a certain level of refinement on the grid, this can be explained by the particularity of the function adopted and because it is a test so severe where no analysis can be performed by the series of Taylor. QUICK, UNIFAES and LOADS again were the schemes that presented the best performance in this test, besides having superior accuracy.



(a) Exact and numerical profiles for a stepwise profile. (b) Numerical RMS error dependence on grid refinement.

Figure 9. Analysis of a stepwise profile.

4. CONCLUSIONS

With the increase of λ/Pe ratio the errors of all schemes grows to the point where there are practically no more difference between the schemes at high frequencies. During this increase regions with low numerical errors are formed for some schemes. The family of exponential-type schemes as UNIFAES, LOADS and Simple Exponential alongside the QUICK, presented the best performance for ratios of $\lambda/Pe < 0.5$. In the analysis of ratios $\lambda/Pe > 0.5$, for solutions CD e DC, the central, QUICK and UNIFAES schemes present the best performance, the central due to the favoring of the zone of the ratio in which the analysis occurred. It was evidenced that the performance of the schemes depends on the region where λ/Pe ratio is located and the type of elementary solution imposed as the dirichilet condition.

The schemes that present the best performance in most test cases are QUICK, UNIFAES and LOADS, however, first QUICK may present oscillatory behavior in its convergence, second LOADS scheme is very overloaded in its formulation and besides presenting oscillatory behavior. The UNIFAES scheme is always among the best schemes, when it is not the best. He has been stable in all the tests carried out in this paper studying the influence of high relations of the ratios of λ/Pe with the increase of eigenvalue. In addition to that, the results presented in Nascimento *et al.* (2018) showed that the UNIFAES have the lowest dependence on the direction of the flow besides being unconditionally stable for any test performed with high Peclet numbers without oscillatory behavior.

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