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Theoretical analyze of turbulent channel flow with thermal effects - the influence of the turbulent Prandtl number

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Abstract. *The authors, on the present paper, aims to provide a study on the influence of the turbulent Prandtl number on a turbulent flow thermal analysis. Such parameter is used to model the turbulent thermal diffusion on the energy equation and has been an academic challenge for decades. On these regards, on the present work, the efforts were focused on solving this problem and studying the influence of the turbulent Prandtl number on the results. For the solution, the unidimensional Navier-Stokes and energy equations were solved. The methodology applied to the differential equations were the Reynold average Navier-Stokes method (RANS), that simplifies the adoption of a time averaged approach, that truncate advective terms and justify permanent regimes. The problem was defined as a Poiseuille plane channel flow. Results were then compared with DNS for validation and used to adjust the turbulent Prandtl number as a function of the turbulent Reynolds number searching optimization in terms of accuracy. Results shown that is possible, with such method, to reach an accuracy even bigger then with the turbulent Prandtl numbers provided by the DNS solution, as it is capable of compensate other approximations of the model's simplification.*

Keywords: *turbulent Prandtl number, Poiseuille plane channel, Reynolds averaged Navier Stokes (RANS), turbulent viscosity, turbulence.*

1. INTRODUCTION

Simulate turbulent flows is a complex task. It involve's nonlinear behavior and, in most cases, can not be solved algebraically. Such complexity is well described on John C. Sommerer (1997), where it was said that the chaotic and continuum aspects of the problem make it such a difficult subject to study. With that in mind, simplification methods are a constant concern on this field as the numeric solutions for a large number of problems require big computational efforts. In these regards, on the present paper, a unidimensional RANS (Reynolds Averaged Navier-Stokes) methodology was prepared to solve the temperature field on Poiseuille turbulent channel flows (Poiseuille, 1846), with the focal objective of providing an accurate but more viable mathematical model concerning the problem.

To solve the temperature field, the thermal energy equation was pondered with convection phenomenon. For that, the velocity field had to be solved, what brings the Navier-Stokes equation for the mathematical approach. It was necessary the modeling of the advective nonlinear terms on the expressions to simplify these equations and close the model. Experimental studies like Nikuradse (1966), Van Driest (1956), conceptual hypothesis, as Boussinesq (1877), and pure computational methods, like numeric adjustment for the turbulent Prandtl number were used. Details are presented and discussed on the present paper.

This theoretical approach was notable as it got the results closer to the DNS solutions (Kawamura, 2000), (Kawamura, 2007), than when the turbulent Prandtl number was determined with data from these DNS. What implies that the adjustment compensated even the error from the numerical resolution of the velocity field, and other approximations from the simplified approach for thermal effects.

2. Methodology

The channel was defined as infinite on the x axis, with a certain width on the y axis and infinite depth, as described on Fig.1. So the system was a flow between two infinity plates. The physical properties was imposed as a Newtonian, incompressible and turbulent flow for a Prandtl number of 0.71. A constant pressure gradient was imposed on the stream wise direction. On the walls, a constant temperature gradient was imposed, with a constant thermal energy flux. Than, the energy and Navier-Stokes equations were developed with respect to these conceptual considerations.

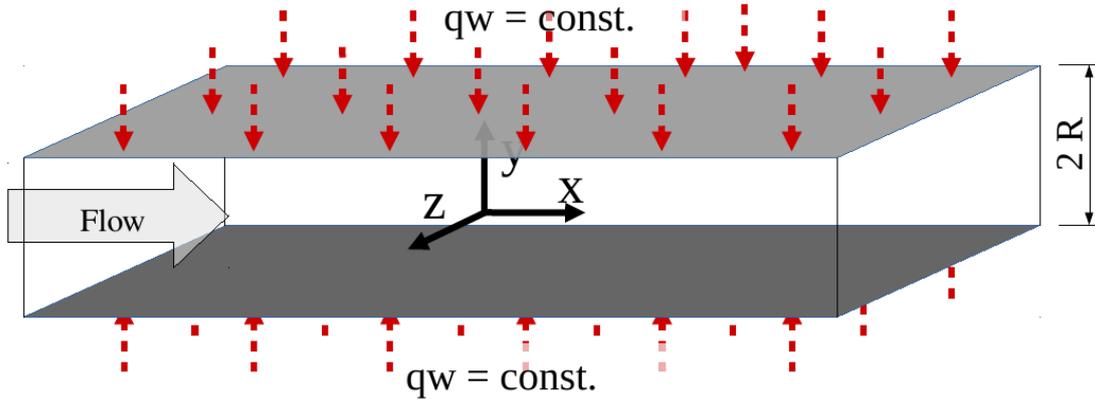


Figure 1. Representation of the channel flow.

2.1 Differential mathematical model

Initially, there was the energy balance equation in a convective context for a statistical permanent regime. The fully developed flow argument allows to truncate the time dependency, what resulted on Eq.(1):

$$\vec{\nabla} \cdot (\rho C_p T \vec{V}) = \vec{\nabla} \cdot (k \vec{\nabla} T). \quad (1)$$

The equation was then developed with the mean values decomposition, that is, $f(x) = \bar{f}(x) + f'(x)$. Fluctuations on the walls were considered to be null, same to the mean velocity on the normal to the flow directions, what resulted on Eq.(2).

$$\frac{\partial}{\partial x} (\overline{T'u'}) + \frac{\partial}{\partial x} (\bar{u}\bar{T}) + \frac{\partial}{\partial y} (\overline{T'v'}) + \frac{\partial}{\partial y} (\bar{v}\bar{T}) = \frac{\partial}{\partial x} \left(\alpha \frac{\partial \bar{T}}{\partial x} \right) + \frac{\partial}{\partial y} \left(\alpha \frac{\partial \bar{T}}{\partial y} \right). \quad (2)$$

Then, it was noted that, even for a fully developed flow, the temperature field does not reach a developed condition. With that in mind, other strategy was required to be formulated. With constant heat from the walls, the temperature was increasing and tending to infinity in the x direction, presenting linear behavior (H. Kawamura, 1999). The argument of linearity then made it possible to define a new variable, that would stay constant for a fully developed flow. The temperature difference $T^*(y) = T(x, y) - T_w(x)$ where $T_w(x)$ was the wall temperature. This approach granted a similarity effect that resumed the problem to a one dimensional system (Fig.2).

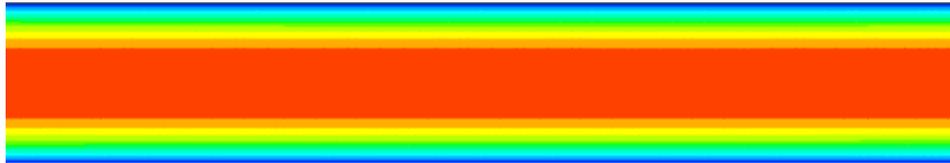


Figure 2. Difference of temperature field over a channel.

Then, the Eq.(2) with this new variable was developed:

$$\begin{aligned} \frac{\partial}{\partial x} (\overline{T_w'u'}) + \frac{\partial}{\partial x} (\overline{T^*u'}) + \frac{\partial}{\partial x} (\bar{u}\bar{T}^*) + \frac{\partial}{\partial x} (\bar{u}\bar{T}_w) + \frac{\partial}{\partial y} (\overline{T^*v'}) + \\ \frac{\partial}{\partial y} (\overline{T_wv'}) + \frac{\partial}{\partial y} (\bar{v}\bar{T}^*) + \frac{\partial}{\partial y} (\bar{v}\bar{T}_w) = \frac{\partial}{\partial x} \left(\alpha \frac{\partial (\overline{T^* + T_w})}{\partial x} \right) + \frac{\partial}{\partial y} \left(\alpha \frac{\partial (\overline{T^* + T_w})}{\partial y} \right). \end{aligned} \quad (3)$$

Then many terms were truncated due the physical nature of the problem. Then it resulted in Eq.(4).

$$\frac{\partial}{\partial y} \left(\alpha \frac{\partial \bar{T}^*}{\partial y} - (\overline{T^*v'}) \right) = \bar{u} \frac{\partial \bar{T}_w}{\partial x}. \quad (4)$$

Where $\frac{\partial \bar{T}_w}{\partial x}$ represents the temperature gradient in the x direction.

To develop the Turbulent flux (the mean value of the product between temperature difference fluctuation and the velocity fluctuation on y axis) a closure model was required. For that, the Boussinesq hypothesis analogy was used. With the same analogy to the Fourier model for thermal molecular diffusive transport, it is possible to write the thermal analogy to the Boussinesq model (Boussinesq, 1877), describing a turbulent thermal diffusivity, as presented on Eq.(5):

$$-\overline{(u'v')} = \nu_\tau \frac{\partial \bar{u}}{\partial y} \implies -\overline{(T'^*v')} = \alpha_\tau \frac{\partial \bar{T}^*}{\partial y}. \quad (5)$$

Then, replacing Eq.(4), with some algebraical simplifications, Eq.(6) was obtained:

$$\frac{\partial}{\partial y} \left((\alpha + \alpha_\tau) \frac{\partial \bar{T}^*}{\partial y} \right) = \bar{u} \frac{\partial \bar{T}_w}{\partial x}. \quad (6)$$

The turbulent Prandtl number was then introduced, as $Pr_\tau = \frac{\nu_\tau}{\alpha_\tau}$. It was necessary to bring the turbulent viscosity that already had a closure model established as described on Antonialli (2015). Then, Eq. (6) originated Eq. (7):

$$\frac{\partial}{\partial y} \left(\left(\alpha + \frac{\nu_\tau}{Pr_\tau} \right) \frac{\partial \bar{T}^*}{\partial y} \right) = \bar{u} \frac{\partial \bar{T}_w}{\partial x}. \quad (7)$$

The turbulent viscosity had to be modeled. For that, the Prandtl's mixing length model (Silveira Neto, 2018) was used. Mathematically it can be described as Eq. (8):

$$\nu_\tau = l_m^2 \left| \frac{\partial \bar{u}}{\partial y} \right|. \quad (8)$$

So, the mixing length (l_m) had to be modeled. For that, the variable mixing length model was used. Although the constant value of 0.14 could provide good results for free shear flows, on viscous channel flows it was generating inconsistent results. It was happening because, near the walls, the molecular effects are much bigger than the turbulent. To bypass the problem, it was used the Nikuradse experimental model (Nikuradse, 1966), as it become null near the wall, as on Eq. (9):

$$L \left(\frac{y}{R} \right) = \frac{l_m}{R} = 0.14 - 0.08 \left(\frac{y}{R} \right)^2 - 0.06 \left(\frac{y}{R} \right)^4. \quad (9)$$

To further enhance the model, the Van Driest's studies (Van Driest, 1956) together with the generalization of Cebeci and Bradshaw (Cebeci, 1984) were used. Mathematically, the resulting expression then was given by Eq. (10):

$$L \left(\frac{y}{R} \right) = \frac{l_m}{R} = \left(0.14 - 0.08 \left(\frac{y}{R} \right)^2 - 0.06 \left(\frac{y}{R} \right)^4 \right) \left\{ 1 - e^{[(\frac{y}{R}-1) \frac{Re_\tau}{26}]} \right\}. \quad (10)$$

This formulation was substituted on Eq. (7). Worth mention that $\left| \frac{\partial \bar{u}}{\partial y} \right| = -\frac{\partial \bar{u}}{\partial y}$, as $\frac{\partial \bar{u}}{\partial y}$ is negative for positive values of y. Considering the problem's symmetry, it was solved considering the domain $0 \leq y \leq R$:

$$\frac{\partial}{\partial y} \left(\left(\alpha - \frac{R^2 L^2 \left(\frac{y}{R} \right)}{Pr_\tau} \frac{\partial \bar{u}}{\partial y} \right) \frac{\partial \bar{T}^*}{\partial y} \right) = \bar{u} \frac{\partial \bar{T}_w}{\partial x}. \quad (11)$$

To understand the relationship that exists between the streamwise gradient of temperature $\frac{\partial \bar{T}_w}{\partial x}$ and the velocity, an energy balance analysis was necessary. For this, it was used the convective Thermal energy flux equation Eq. (12):

$$q_{conv.} = \dot{m} C_p \Delta T_m. \quad (12)$$

Studying the boundaries conditions of the system, as on Fig.1, it was noticed that the resulting conduction thermal energy was not null, therefor it had to be related to the convective thermal energy flux to generate a relative steady condition. Then, these measures, for a permanent regime, must had to be the same:

$$2q_w b \Delta x = \dot{m} C_p \Delta T_m, \quad (13)$$

where b was the depth of the channel. So substituting the mass flux as $\dot{m} = u_m 2Rb\rho$, and making the temperature difference to be expressed as a function of the temperature gradient $\Delta T_m = \frac{\partial \overline{T_w}}{\partial x} \Delta x$:

$$u_m \frac{\partial T_m}{\partial x} = \frac{q_w}{R\rho C_p}, \quad (14)$$

where u_m was the mean velocity on the vertical channel section, and T_m was the bulk temperature. To associate this equality to the main equation, it is important to remember that all the temperature field varies linearly on the streamwise direction. So is correct to assume that $\frac{\partial(T_m)}{\partial x} = \frac{\partial(\overline{T_w})}{\partial x}$, therefore:

$$\frac{\partial \overline{T_w}}{\partial x} = \frac{q_w}{R\rho C_p u_m}. \quad (15)$$

Substituting Eq.(15) on Eq.(11):

$$\frac{\partial}{\partial y} \left(\left(\alpha - \frac{R^2 L^2 (\frac{y}{R})}{Pr_\tau} \frac{\partial \overline{u}}{\partial y} \right) \frac{\partial \overline{T^*}}{\partial y} \right) = \frac{\overline{u}}{u_m} \frac{q_w}{R\rho C_p}. \quad (16)$$

Then it was nondimensionalized to facilitate comparison with the DNS data. For that purpose the variables were replaced as follows: $\tilde{y} = \frac{y \cdot Re_\tau}{R}$, $\tilde{u} = \frac{\overline{u}}{u_\tau}$, $\tilde{T} = \frac{\overline{T}}{T_\tau}$, $Re_\tau = \frac{u_\tau R}{\nu}$, $Pr_\tau = \frac{\nu_\tau}{\alpha_\tau}$, $Pr = \frac{\nu}{\alpha}$ and $T_\tau = \frac{q_w}{\rho C_p u_\tau}$.

$$\frac{\partial}{\partial \tilde{y}} \left(\left(\frac{Re_\tau}{Pr} - \frac{L^2 (\frac{\tilde{y}}{Re_\tau})}{Pr_\tau} Re_\tau^3 \frac{\partial \tilde{u}}{\partial \tilde{y}} \right) \frac{\partial \tilde{T}^*}{\partial \tilde{y}} \right) = \frac{\tilde{u}}{u_m}. \quad (17)$$

Equation (17) represents the final form of the differential model. Solve exactly such equation was impossible. The solution then was accomplished using a numerical method.

To solve the velocity field, a RANS methodology was applied too, as described on Antonialli (2015). From this author an exact solution for $\frac{d\tilde{u}}{d\tilde{y}}$ was obtained. The equation can be described as:

$$\frac{d\tilde{u}}{d\tilde{y}} = - \frac{2\tilde{y} \frac{1}{Re_\tau}}{1 + \sqrt{1 + 4L^2 (\frac{\tilde{y}}{Re_\tau}) Re_\tau^2 \tilde{y}}}. \quad (18)$$

So, with the equations obtained (Eq. (17) and Eq. (18)), it was necessary to solve both numerically.

2.2 Numerical procedure

To solve the differential equations, an Eulerian net of discrete cells were formulated. The last cell center was positioned on the wall, and a mid-cell face was positioned on the central line of the channel. The Runge-kutta method was applied to the velocity field and central differences was applied to the temperature, being the results of the momentum equation transported to the thermal energy equation. DNS data from Kawamura (2000) and Kawamura (2007) were used for validation. The turbulent Reynolds numbers analyzed were 150, 395, 640 and 1020, that correspond respectively to the global Reynolds numbers of 4560, 14062, 24428 and 41441.

The turbulent Prandtl number was a parameter to be modeled too. A quick experimentation with the value of this parameter shown that the results were very sensitive to changes on it. Than the idea of adjust this parameter was adopted. It consisted in, by trying and error, experimenting a variety of turbulent Prandtl values, aiming to an error, for each try, smaller and smaller, until a given prescribed error for the parameter (two decimal places).

The value of the optimized Pr_τ was depending on the Reynolds number. With that in mind, was made the same adjust for other Reynolds, and a linearization on the results, to create a model of turbulent Prandtl number as a function of the turbulent Reynolds number.

2.3 Prandtl analysis

First, for the turbulent Reynolds number of 1020, it was found a value for the turbulent Prandtl number that provide minimum error. This was $Pr_{\tau} = 0.905$, for a L2 norm of 0.1515, on Fig.3. But when applied on other turbulent Reynolds numbers, like 150, 395 and 640, the L2 norm of the results were respectively 0.348, 0.239 and 0.193. What shows the increasing inaccuracy of the method for low turbulent Reynolds number cases. This inaccuracy was, in parts, a consequence of the decreasing turbulence for low Reynolds numbers, what escaped from the conceptual delimitations (a turbulent domain). The turbulent Prandtl number has Reynolds dependence too, so determine it as a constant was resulting in more error.

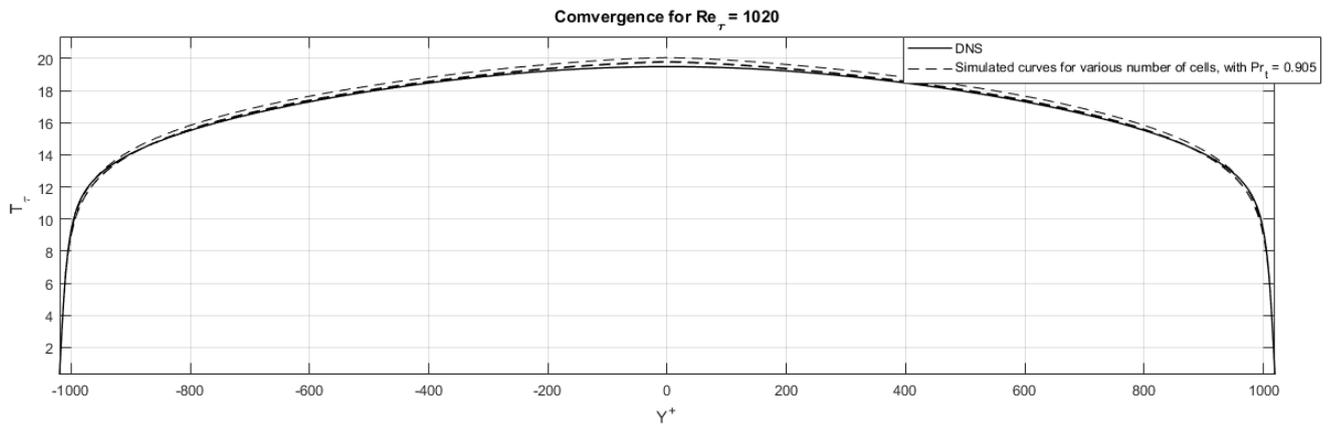


Figure 3. Simulations for various numbers of space cells for $Re_t = 1020$.

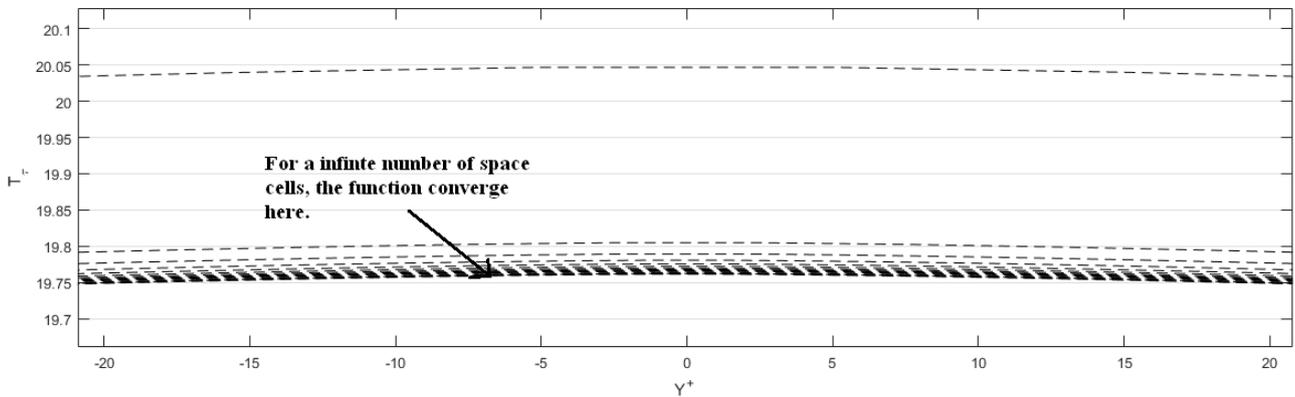


Figure 4. Zoom on the graph of simulations for various numbers of space cells for $Re_t = 1020$.

When studied each turbulent Reynolds number, other values of Pr_{τ} were obtained. For the turbulent Reynolds numbers of 150, 395 and 640, the best values of Pr_{τ} were respectively 0.944, 0.905 and 0.905. With such information, an approximated model was proposed to result on a value to the turbulent Prandtl number for each Reynolds number. Mathematically it can be expressed as Eq. (19):

$$Pr_{\tau} = -3.734 * 10^{-10} * Re_{\tau}^3 + 7.674 * 10^{-7} * Re_{\tau}^2 - 0.0004886 * Re_{\tau} + 1.001 \quad (19)$$

Results of the model can be seen ahead, for all the mentioned Turbulent Reynolds numbers.

3. Results and discussion

For turbulent Reynolds numbers of 150, 390, 640 and 1020, the results in comparison with the DNS can be seen on Fig. 5, Fig. 6, Fig. 7 and Fig. 8.

The accuracy came very close to the DNS solution, but presenting a very small computational cost. This fact shows that a semi-analytical method can provide optimal results, even when compared to the most heavy and rigid methods of CFD. The adjusted turbulent Prandtl number proved to be an efficient method to close the model and can be used in other works of plane turbulent Poiseuille flows in a very simple and cost effective way.

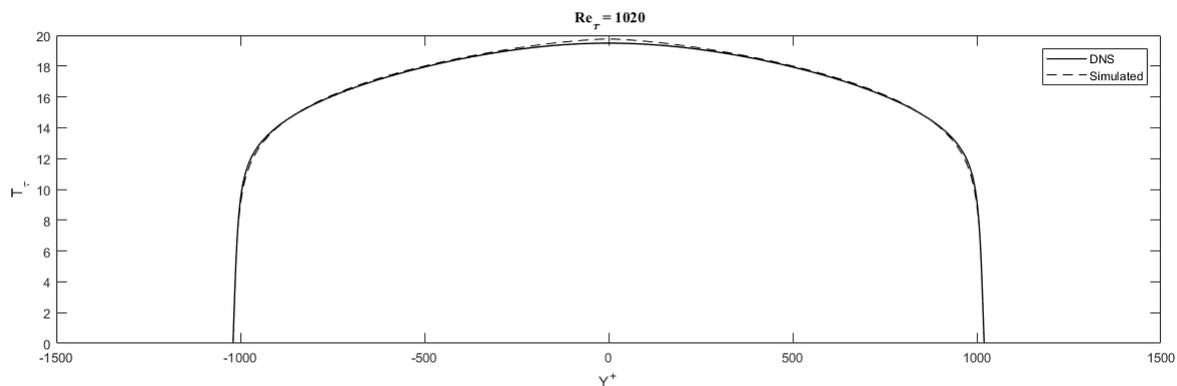


Figure 5. Comparison between DNS and simulation for $Re_t = 1020$. Norm L2 = 0.1515

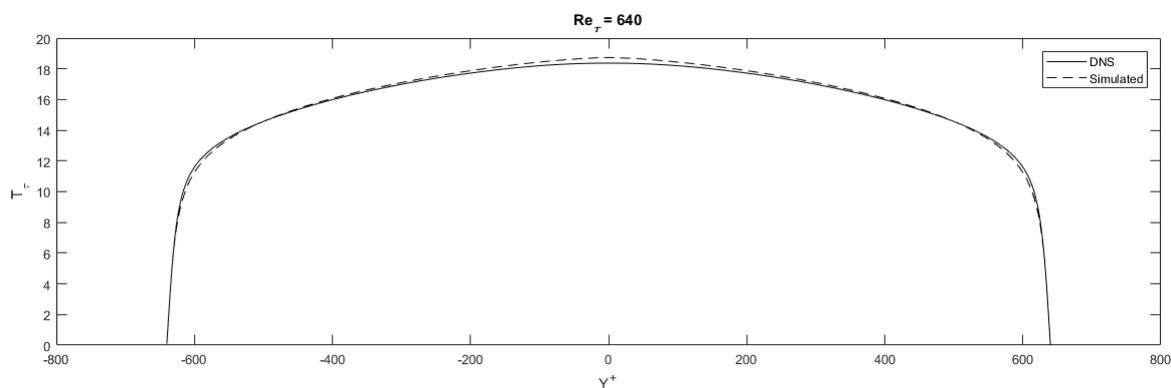


Figure 6. Comparison between DNS and simulation for $Re_t = 640$. Norm L2 = 0.1930

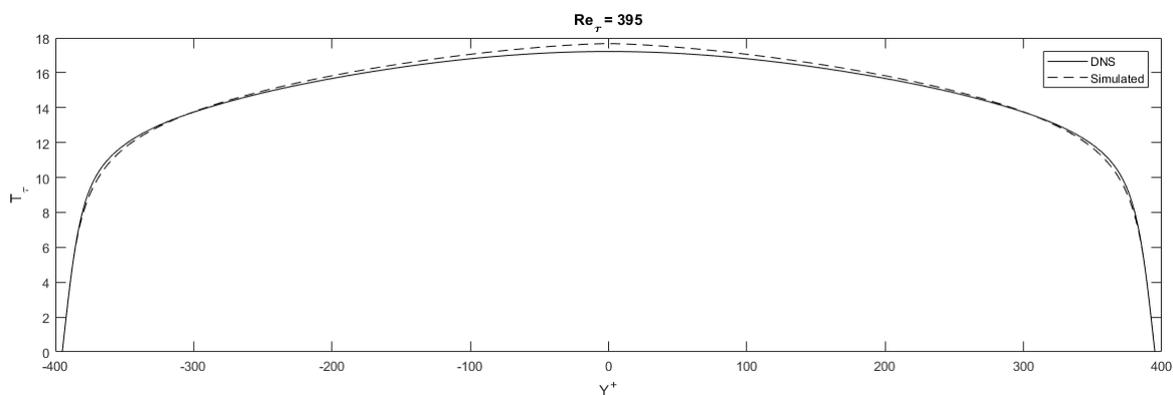


Figure 7. Comparison between DNS and simulation for $Re_t = 395$. Norm L2 = 0.2162

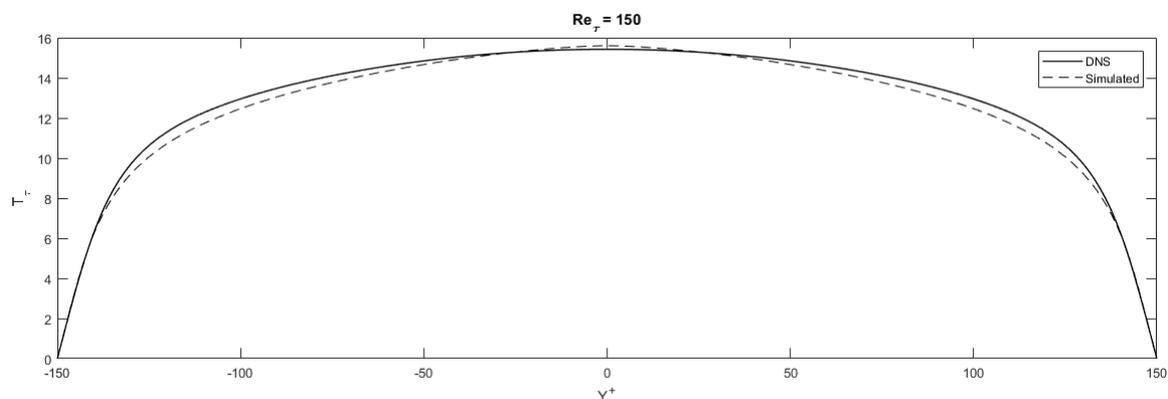


Figure 8. Comparison between DNS and simulation for $Re_t = 150$. Norm L2 = 0.2512

To use the turbulent Prandtl number from the DNS, an averaged value of the domain had to be made for each turbulent Reynolds number, as this parameter varies with the distance from the wall. So, the values obtained were 0.9501, 0.8519 and 0.8505 respectively to the Reynolds numbers of 150, 390 and 640. So, as it can be seen on Tab.1, the errors are bigger than the originated from the meta model.

Table 1. L2 norm from the simulations with the turbulent Prandtl number from the meta model and from the DNS.

Turbulent Reynolds number	Meta model	DNS
150	0.2512	0.2662
395	0.2162	0.3988
640	0.1930	0.4355
1020	0.1515	— ⁽¹⁾

⁽¹⁾ there was no DATA from DNS.

4. CONCLUSION

The semi exact model to solve the temperature domain on a Poiseuille channel flow was appropriated to study the turbulent Prandtl number. This parameter has big influence on the results and can provide opportunities of meta modeling. With the appropriated value to this parameter, gains on the accuracy were notable. First the errors were 0.348, 0.239 and 0.193 respectfully for the turbulent Reynolds of 150, 395 and 640, being 0.2512, 0.2162, 0.1930 after the meta model. So this kind of approach can be of great value for thermal analysis.

In terms of academic growth, the present paper has a heavy educational potential. It address a big number of theoretical definitions that are the basis of fluid-dynamics.

5. ACKNOWLEDGEMENTS

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