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FINITE ELEMENTS SIMULATION OF HEAT TRANSFER IN THIN PLATES WITH TEMPERATURE DEPENDENT CONDUCTIVITY

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Abstract. *This article studies the steady-state heat transfer in a thin flat plate with temperature dependent thermal conductivity. The thin plate assumption allows a two-dimensional approach using the mean value of the temperature over the plate thickness. The problem considers a known non-uniform internal heat supply and convective heat exchange between the environment and the plate, giving rise to a nonlinear partial differential equation subjected to a Neumann boundary condition. A piecewise constant function of the temperature is assumed for the thermal conductivity. In the sequence, a Kirchhoff transformation is used to build a convenient equivalent problem with an equivalent minimum principle. This minimization problem is numerically simulated using a finite element approach with triangular elements. Some results show that the employed methodology deals efficiently with the problem. This is particularly relevant when the thermal conductivity decreases as the temperature increases.*

Keywords: conduction heat transfer, flat plate, Kirchhoff transformation, functional minimization, finite element formulation.

1. INTRODUCTION

In many relevant engineering problems thermal conductivity dependency on temperature cannot be neglected. As an example, in the case of crystalline ropes of single-walled carbon nanotubes, the thermal conductivity decreases smoothly with decreasing temperature displaying a linear temperature dependency below 30 K (Hone et al., 1999). In applications in optoelectronics, Geseley et al. (1997) verified that the thermal conductivity increases with temperature increase. The maximum thermal conductivities of the polycrystalline zinc oxide (an important material for optoelectronics) occur at about 60 K and their values are almost an order of magnitude lower than bulk ZnO (Alvarez-Quintana et al., 2010).

This paper presents finite element simulations for steady-state heat conduction in a thin flat plate with temperature dependent thermal conductivity, an internal non-uniform heat supply and convective heat transfer at the plate boundaries. Considering a two-dimensional description (thin plate hypothesis), the mathematical model consists of a nonlinear partial differential equation subjected to a Neumann boundary condition. Due to the problem nonlinearity, a Kirchhoff transformation is employed and, in the sequence, an equivalent minimum principle for the transformed problem (Saldanha da Gama et al., 2017) is obtained. The minimization of the functional is carried out by means of a piecewise linear continuous approximation (triangular finite elements). The Conjugate Gradient Method is employed, with the help of the MathLab software, in order to achieve the approximation in each node.

2. MECHANICAL MODEL

Steady-state heat transfer process in a flat plate represented in rectangular Cartesian coordinates x, y and z, defined by the bounded open set $\Omega \equiv \{(x,y,z)$ such that $(x,y)\in\Gamma$ and $0<z<L\}$, where the positive constant L is small when compared with the mean lengths along the x and the y axes and the plate is sufficiently thin for neglecting the heat transfer from/to the boundary of Γ and assuming a convective heat exchange from/to the subsets of $\partial\Omega$ in which $z=0$ and $z=L$ is described by:

$$
\begin{aligned}
& div\left(k\, grad\, T\right) + \dot{q} = 0 \quad \text{in } \Omega \\
& \left(-k\, grad\, T\right)\cdot \mathbf{n} = 0 \quad \text{on } \partial\Gamma \\
& \left(-k\, grad\, T\right)\cdot \mathbf{k} = h_L\left(T - T_{L\infty}\right) \quad \text{at } z = L \\
& \left(k\, grad\, T\right)\cdot \mathbf{k} = h_0\left(T - T_{0\infty}\right) \quad \text{at } z = 0
\end{aligned}
\tag{1}
$$

In the mathematical description stated in Eq. (1) h_0 and h_L represent the convection heat transfer coefficients, assumed constants, $T_{0\infty}$ and $T_{L\infty}$ represent the temperatures of the environment above and below the plate, $\mathbf{n}$ is the unit outward normal defined on $\partial\Gamma$, $\dot{q}$ is the internal heat supply (per unit time and per unit volume), k is the thermal conductivity; a positive valued temperature dependent quantity $k = \hat{k}\left(T\right) > 0$.

Equation (1) is equivalent to

$$
\begin{aligned}
& \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = 0, \quad \left(x, y\right)\in\Gamma,\ 0 < z < L \\
& -k\left(\frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j}\right)\cdot\mathbf{n} = 0, \quad \left(x, y\right)\in\partial\Gamma,\ 0 < z < L \\
& k\frac{\partial T}{\partial z} = h_0\left(T - T_{0\infty}\right), \quad \left(x, y\right)\in\Gamma,\ z = 0 \\
& -k\frac{\partial T}{\partial z} = h_L\left(T - T_{L\infty}\right), \quad \left(x, y\right)\in\Gamma,\ z = L
\end{aligned}
\tag{2}
$$

Integrating Eq. (2) over the plate thickness, considering the boundary conditions at $z = 0$ and $z = L$, and defining $\bar{T}$ as the mean temperature value for a given position $\left(x, y\right)$ and $\bar{q}$ the mean value of the internal heat source $\dot{q}$ evaluated over the z axis as:

$$
\bar{T} = \frac{1}{L}\int_0^L \hat{T}\left(x, y, z\right)dz, \quad T = \hat{T}\left(x, y, z\right) \qquad \bar{q} = \frac{1}{L}\int_0^L \dot{q}\,dz
\tag{3}
$$

So that the steady-state heat transfer process in a flat plate may be expressed as:

$$
\begin{aligned}
& \frac{\partial}{\partial x}\left(k\frac{1}{L}\left(\frac{\partial}{\partial x}\int_0^L T dz\right)\right) + \frac{\partial}{\partial y}\left(k\frac{1}{L}\left(\frac{\partial}{\partial y}\int_0^L T dz\right)\right) + \\
& \qquad + \frac{1}{L}\left[-h_L\left(T\big|_{z=L} - T_{L\infty}\right) - h_0\left(T\big|_{z=0} - T_{0\infty}\right)\right] + \bar{q} = 0, \quad \left(x, y\right)\in\Gamma \\
& -k\left(\frac{1}{L}\left(\frac{\partial}{\partial x}\int_0^L T dz\right)\mathbf{i} + \frac{1}{L}\left(\frac{\partial}{\partial y}\int_0^L T dz\right)\mathbf{j}\right)\cdot\mathbf{n} = 0, \quad \left(x, y\right)\in\partial\Gamma
\end{aligned}
\tag{4}
$$

A thin plate is such that $\bar{T} = \frac{1}{L}\int_0^L T dz \cong T\big|_{z=0} \cong T\big|_{z=L}$, so that Eq. (4) may be rewritten as

$$
\begin{aligned}
& \frac{\partial}{\partial x}\left(\bar{k}\frac{\partial \bar{T}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\bar{k}\frac{\partial \bar{T}}{\partial y}\right) - \bar{T} + \gamma = 0, \quad \left(x, y\right)\in\Gamma \\
& \qquad \left(\frac{\partial \bar{T}}{\partial x}\mathbf{i} + \frac{\partial \bar{T}}{\partial y}\mathbf{j}\right)\cdot\mathbf{n} = 0, \quad \left(x, y\right)\in\partial\Gamma \\
& \text{with} \quad \bar{k} = \frac{kL}{h_L + h_0} = \hat{\bar{k}}\left(\bar{T}\right) \qquad \gamma = \frac{h_L T_{L\infty} + h_0 T_{0\infty} + \bar{q}L}{h_L + h_0} = \hat{\gamma}\left(x, y\right)
\end{aligned}
\tag{5}
$$

It is important to note that the thermal conductivity $\bar{k}=\hat{\bar{k}}\left(\bar{T}\right)$ depends on the mean temperature value $\bar{T}$, while $\gamma=\hat{\gamma}\left(x,y\right)$ depends on the spatial position $\left(x,y\right)$.

Now the Kirchhoff transformation $K\left(\bar{T}\right)$ is introduced in terms of a variable ω (Arpaci, 1966):

$$K\left(\bar{T}\right)=\omega=\frac{L}{h_L+h_0}\int_{T_R}^{\bar{T}}\hat{k}\left(\xi\right)d\xi=\int_{T_R}^{\bar{T}}\hat{\bar{k}}\left(\xi\right)d\xi \tag{6}$$

in which T_R is a conveniently chosen reference temperature. Using the definition stated in Eq. (6), the heat transfer process in a thin plate (Eq. 5)), yields:

$$\begin{aligned}&\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}-K^{-1}\left(\omega\right)+\gamma=0,\quad\left(x,y\right)\in\Gamma\\&\left(\frac{\partial\omega}{\partial x}\mathbf{i}+\frac{\partial\omega}{\partial y}\mathbf{j}\right)\cdot\mathbf{n}=0,\quad\left(x,y\right)\in\partial\Gamma\end{aligned} \tag{7}$$

The existence of $K^{-1}\left(\omega\right)$, for any value of ω, is always ensured if there exists a constant $\delta>0$ such that $\hat{k}\left(\bar{T}\right)>\delta>0$ for any temperature $\bar{T}$ (Saldanha da Gama et al., 2017).

As an example, consider the thermal conductivity a piecewise constant function of the temperature $\bar{T}$ that could derive from actual data involving the temperature and the corresponding conductivity for a given material (Saldanha da Gama et al., 2017):

$$\bar{k}=\begin{cases}\bar{k}_1 & for\quad \bar{T}_1\leq\bar{T}\\ \bar{k}_2 & for\quad \bar{T}_2\geq\bar{T}>\bar{T}_1\\ \bar{k}_3 & for\quad \bar{T}>\bar{T}_2\end{cases} \tag{8}$$

Giving rise to:

$$\omega=K\left(\bar{T}\right)=\frac{\bar{k}_1\left(\bar{T}-\bar{T}_1\right)}{2}+\frac{\bar{k}_3\left(\bar{T}-\bar{T}_2\right)}{2}+\left(\frac{\bar{k}_2-\bar{k}_1}{2}\right)\left|\bar{T}-\bar{T}_1\right|+\left(\frac{\bar{k}_3-\bar{k}_2}{2}\right)\left|\bar{T}-\bar{T}_2\right|+\frac{\bar{k}_2\left(\bar{T}_2-\bar{T}_1\right)}{2} \tag{9}$$

So, the heat transfer process in a thin plate (Eq. (5)) is represented as:

$$\begin{aligned}&\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}-\bar{T}_1-\frac{\left(\omega-\omega_1\right)}{2\bar{k}_1}-\frac{\left(\omega-\omega_2\right)}{2\bar{k}_3}-\left(\frac{1}{2\bar{k}_2}-\frac{1}{2\bar{k}_1}\right)\left|\omega-\omega_1\right|-\left(\frac{1}{2\bar{k}_3}-\frac{1}{2\bar{k}_2}\right)\left|\omega-\omega_2\right|\\&\qquad-\frac{\left(\omega_2-\omega_1\right)}{2\bar{k}_2}+\gamma=0,\quad\left(x,y\right)\in\Gamma\\&\left(\frac{\partial\omega}{\partial x}\mathbf{i}+\frac{\partial\omega}{\partial y}\mathbf{j}\right)\cdot\mathbf{n}=0,\quad\left(x,y\right)\in\partial\Gamma\end{aligned} \tag{10}$$

This transformed equation (equation (10)) has an equivalent minimum principle. In other words, the solution of Eq. (10) may be reached from the minimization of the functional (according to the proofs of existence and uniqueness presented by Saldanha da Gama et al. (2017)) given by:

$$\begin{aligned}I\left[v\right]=&\frac{1}{2}\int_\Gamma\left(grad\,v\cdot grad\,v\right)dA-\int_\Gamma\left(\gamma-\bar{T}_1\right)v dA+\frac{1}{4\bar{k}_1}\int_\Gamma\left(v-\omega_1\right)^2 dA+\frac{1}{4\bar{k}_3}\int_\Gamma\left(v-\omega_2\right)^2 dA\\&+\left(\frac{1}{4\bar{k}_3}-\frac{1}{4\bar{k}_2}\right)\int_\Gamma\left(v-\omega_2\right)\left|v-\omega_2\right|dA+\left(\frac{1}{4\bar{k}_2}-\frac{1}{4\bar{k}_1}\right)\int_\Gamma\left(v-\omega_1\right)\left|v-\omega_1\right|dA+\frac{\left(\omega_2-\omega_1\right)}{2\bar{k}_2}\int_\Gamma v\,dA\end{aligned} \tag{11}$$

As an illustration of the piecewise constant temperature described by Eq. (8), the curve of thermal conductivity versus absolute temperature is presented in Fig. 1 for silicon with a distinct temperature range — namely from 300 K to 400 K. It is important to note that the choice of *T*1 and *T*2 depends on the expected range for the temperature distribution.

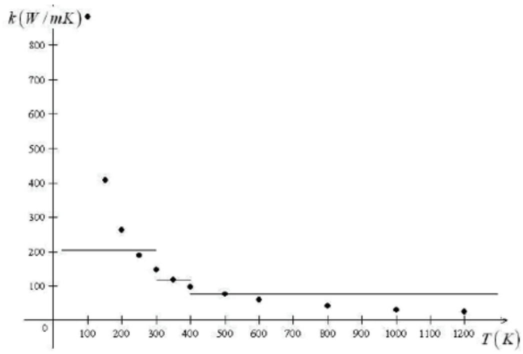


Figure 1. Thermal conductivity vs. absolute temperature for silicon. Piecewise constant approximation for silicon with $T_1 = 300\ K$ and $T_2 = 400\ K$

3. FINITE ELEMENT SIMULATION

The problem described by Eq. (11) is simulated using a finite element approximation (Ciarlet, 1988; Reddy, 1993). A Conjugate Gradient Method was employed within the Matlab software. Conjugate gradients are optimization algorithms, which display strong local and global convergence properties, requiring low memory. Initially these algorithms where restrained to solving linear symmetric positive-definite systems. In the sequence, they were extended to nonlinear unconstrained optimization problems (Golub and O'Leary, 1989).

4. NUMERICAL RESULTS

To following results, show the temperature of a plate with a heat source, $\bar{\bar{q}}_1 = 40\frac{kW}{m^3}$, located in one quarter of its domain, as represented in the gray area shown in Fig. 2.

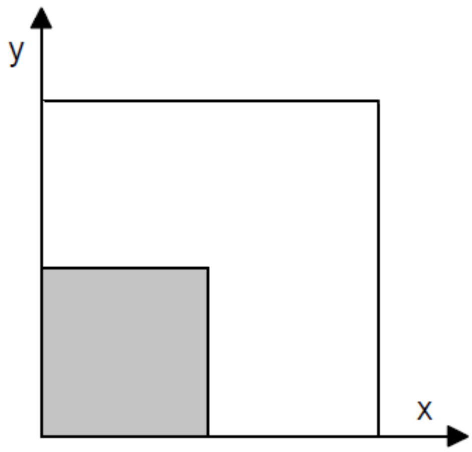


Figure 2. Problem statement.

The following values are considered for the variables in Eq. (9) and (11): $T_{L\infty} = T_{0\infty} = 300\ K$; $h_L = h_0 = 10\frac{W}{m^2.K}$; $\omega_1 = \bar{K}_1\bar{T}_1$ and $\omega_2 = \omega_1 + \bar{k}_2(\bar{T}_2 - \bar{T}_1)$.

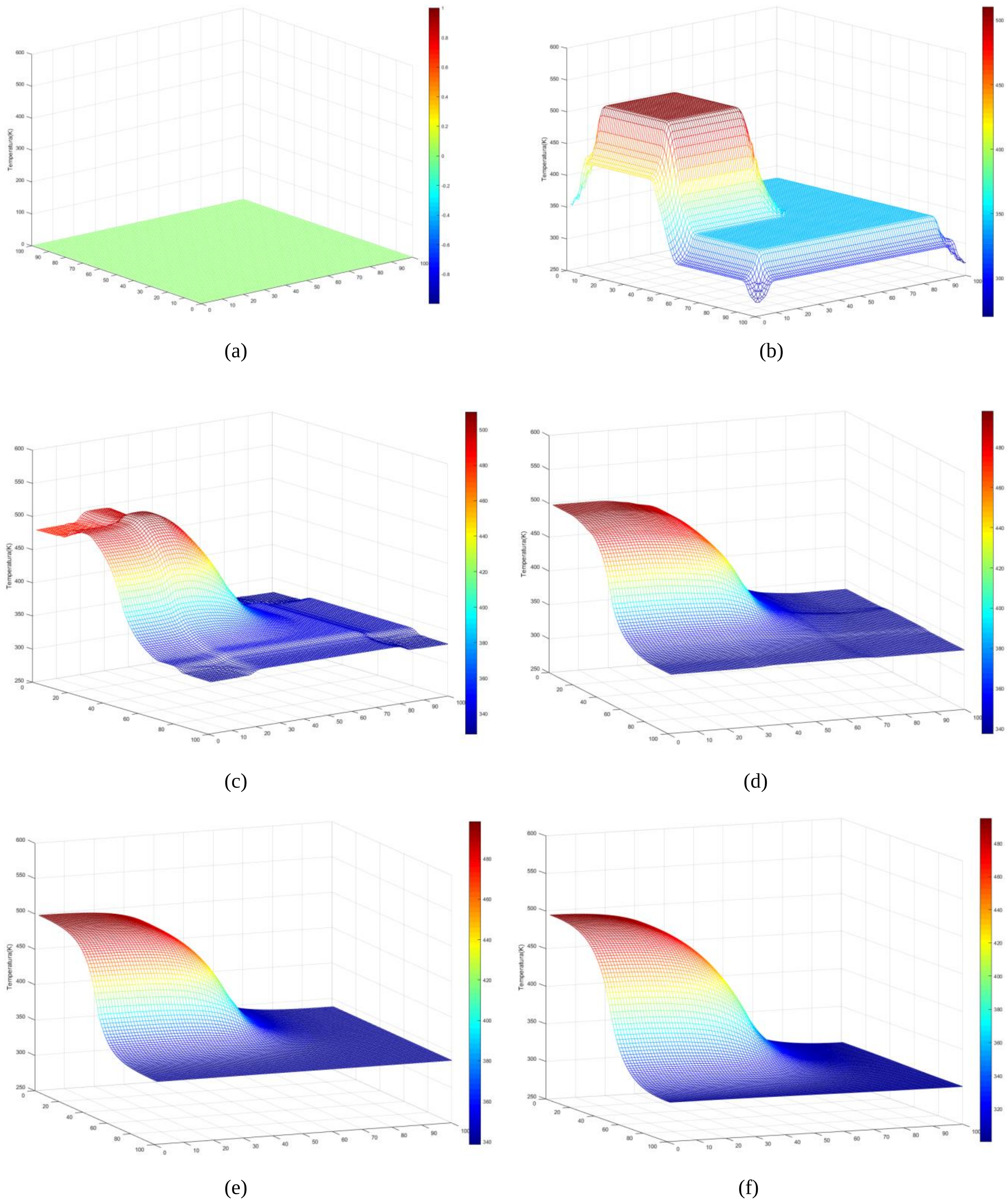
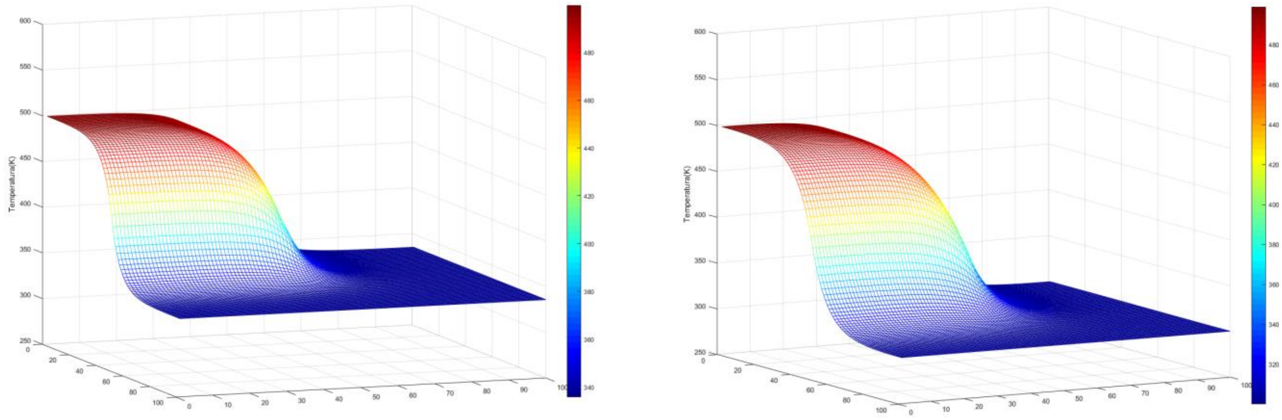


Figure 3. Temperature elevation plots for a Silicon plate in a 100 x 100 finite elements mesh: (a)-(e) temperature-dependent conductivity; (f) average conductivity.

Figure 3 depicts temperature elevation plots considering a silicon plate with a heat source according to Fig. 2. For the temperature-dependent thermal conductivity, the following values are considered for the variables in Eq. (8): $\bar{k}_1 = 210$, $\bar{k}_2 = 130$ and $\bar{k}_3 = 80\,\frac{W}{m.K}$; $\bar{T}_1 = 300\ K$ and $\bar{T}_2 = 400\ K$, while the average conductivity is $\bar{k}_{average} = 130\,\frac{W}{m.K}$.

The intermediate iterations when a piecewise constant approximation for the thermal conductivity is considered are presented in Figs. 3(a)-(d). Fig. 3(a) represents iteration 1 (a), Fig. 3(b) iteration 10, Fig. 3(c) iteration 30 and Fig. 3(d) iteration 50. Fig. 3(d) presents iteration 216, when the function converged. Fig. 3(e) can be compared with Fig. 3(f), when a constant (average) thermal conductivity is considered, in iteration 271.

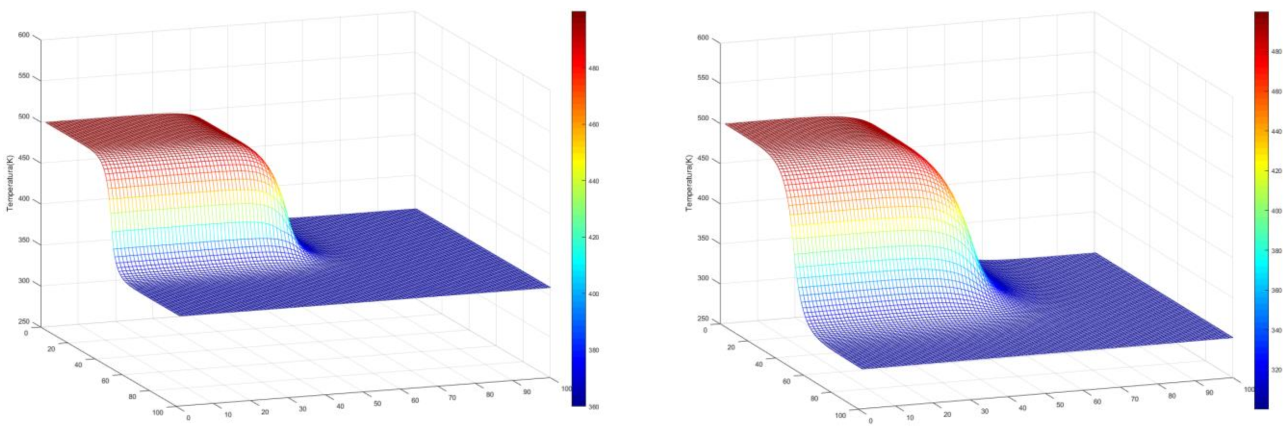
Comparing these two last elevation plots the influence of allowing a piecewise constant dependency of the thermal conductivity on the temperature is clearly noted, one can specifically observe the maximum and minimum temperature values, for instance. This trend is particularly important because in the case of silicon the thermal conductivity decreases as the temperature increases.



(a) (b)

Figure 4. Temperature elevation plots for an Iron plate in a 100 x 100 finite elements mesh: (a) temperature-dependent conductivity; (b) average conductivity.

Figure 4 compares temperature elevation plots considering an iron plate with a heat source according to Fig. 2, with temperature-dependent thermal conductivity (4(a)) and constant (average) thermal conductivity (4(b)). The former considers the following values for the variables in Eq. (8): $\bar{k}_1 = 100$, $\bar{k}_2 = 70$ and $\bar{k}_3 = 45\frac{W}{m.K}$; $\bar{T}_1 = 300\,K$ and $\bar{T}_2 = 400\,K$, while in the latter $\bar{k}_{average} = 70\frac{W}{m.K}$. Convergence is reached in iteration 166 for the temperature-dependent thermal conductivity (4(a)) and in iteration 211 for the constant (average) thermal conductivity (4(b)). Distinct elevation plots are observed in the two cases, particularly concerning the minimum temperatures.



(a) (b)

Figure 5. Temperature elevation plots for an Aluminum Oxide plate in a 100 x 100 finite elements mesh: (a) temperature-dependent conductivity; (b) average conductivity.

Figure 5 (a) represents the piecewise constant temperature-dependent conductivity elevation plots for an Aluminum Oxide plate representing iteration 88, when the function converged. In this case, the following values are considered for the variables in Eq. (8): $\bar{k}_1 = 60$, $\bar{k}_2 = 30$ and $\bar{k}_3 = 12\frac{W}{m.K}$; $\bar{T}_1 = 300\,K$ and $\bar{T}_2 = 400\,K$. In Fig. 5(b) a constant (average) thermal conductivity $\bar{k}_{average} = 30\frac{W}{m.K}$ is considered and convergence is obtained in iteration 142. This last example presents very distinct elevation plots for the two compared cases.

As expected, considering an average (constant) value for the thermal conductivity or considering temperature dependent thermal conductivity has a significant influence in the elevation plots in all the examples considered in this work. This allows concluding that a temperature dependent thermal conductivity should be considered whenever a realistic representation is important, particularly when the thermal conductivity decreases as the temperature increases.

5. FINAL REMARKS

The numerical methodology – namely the minimization of a functional which is equivalent to the transformed heat transfer equation (after the linearization by the Kirchhoff transform) of a thin plate and its subsequent solution by combining a Conjugate Gradient Method and a Finite Element approximation – was able to deal with a realistic thermal conductivity distribution, when the thermal conductivity dependency on temperature cannot be neglected

6. ACKNOWLEDGEMENTS

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