ENERGY FLOW ANALYSIS APPLIED TO DYNAMIC BEHAVIOR OF METAMATERIALS

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Resumo: *Flow of mechanical energy has been used successfully in dynamic analyses of structures. It results in an efficient way of identifying discontinuities due to the notches presence or geometry changes like section or orientation in axial direction of the element. In order to reduce dangerous levels of vibration, many strategies have been applied, among them, one can cite the changes in geometry or material, known Phononic Crystals (PC). One of the newest approach involves the use of elastic metamaterial with the aim of creating vibration bandgaps. This technique uses certain types of geometric or materials configurations, that form a cell that is repeated periodically. It is possible then to obtain its corresponding transfer matrix. In this work, an energy flow analysis is used to study the behavior of a specific elastic metamaterial using Energy Spectral Element Method. Further applications of this behavior are discussed.*
Keywords: *Bandgap, Phononic Cristal, Energy Flow, Metamaterial.*

1. INTRODUCTION

Study of waves in periodic structures has started with Mead in the mid-1970s (Mead, 1970, 1973, 1974, 1975). Recently, some researchers studied the effect on this waves when periodic inclusions are added (Xiao *et al.*, 2012b), change in geometry (Edilson *et al.*, 2016) or material (Xiang and Shi, 2009). Also known as Phononic Crystals or metamaterials, this kind of structure have a "wave filtering" property, thus waves cannot propagate freely through the periodic structures within some frequency ranges, which are called band gaps (Xiao *et al.*, 2012a). Metamaterial when subjected to electromagnetic, acoustic, or elastic waves can show unusual behavior that can be predicted at the design stage. This trait of Phononic Crystals are sometimes created by inclusions or material characteristics that have a smaller wavelength when compared to its base material or geometry. For instance one can cite periodic inclusions of a different materials or cross sections in a bar (Mitchell *et al.*, 2014).

Some researchers describe that the progress in research of dynamic behaviour in periodic structures will lead to a scenario where one can precisely manipulate the frequency bandgaps, in a way that meet its design requirements (Hussein *et al.*, 2014). A investigation of wave propagation and vibration transmission in rod containing multi-degree-of-freedom resonator shows that resonators can create more than one bandgap, they are proportional to the number of degrees of freedom and can increase with frequency (Xiao *et al.*, 2012a). In the work of Liu *et al.* (2000) the Transfer Matrix (TM) method and Bloch theorem are used to investigate phononic crystal (PC) Euler beams on a two parameter foundation Xiao *et al.* (2013). They show that PC Euler beam on this kind of foundation has better vibration isolation characteristics, compared with the homogeneous Euler beam.

The Spectral Element Method - SEM has also been used to model one and two-dimensional structures (Lee, 2009) and was extended as Wave Spectral Element Method - WSEM to model elastic metamaterial rods (Xiao *et al.*, 2012b).

Analysing problems in terms of energy flow is convenient once they have an easy and simple physical interpretation. By the use of energy flow analysis in structures is possible to identify the way the energy propagates. Then one can use this information to make changes in the design so as to minimize the level of vibrational energy. Energy Flow Analyses (EFA) is an analytical tool used to predict structural response in terms of vibrational energy at high- and mid-frequencies (Noiseux, 1970) . This procedure is based on the development of differential equations which have as variables, quantities related to the propagation of energy in simple structural elements such as rods, beams and plates. It is also necessary to derive coupling relations in terms of the quantities dependent on the energy. These quantities are the coefficients of transmission and reflection which represents the transference of energy between different types of joints (rod-rod, beam-beam, plate-plate). In this work equations for flow and density energy, from analytical solution of displacements, are obtained to structural elements like rods. These equations can be obtained from an energy balance and have, as one

application, the prediction of the distribution of power flow and energy density (J.C. Wohlever, 1992; Cho and Bernhard, 1998).

In this work, a study of energy flow on a metamaterial that have two kinds of materials (polymer and steel) is conducted. They are put in an interleaved arrangement along the rod's length. For this case, inclusion material is the steel, that causes a frequency bandgap on the polymer rod. This occurs because of Bragg scattering that happens near frequencies determined by the Bragg conditions $L = n(\lambda/2)$ $(n = 1, 2, 3, ...)$, where L is the lattice constant of the periodic system, λ is the host material wavelength (Xiao *et al.*, 2013) and $k = \pi/L$ is the wavenumber. In this article was used the Spectral Element Method in order to show frequency response on rod using this metamaterial structure. Results are compared with the ones obtained via Transfer Matrix Method (Pilkey, 2002).

2. METAMATERIAL

It is considered an elastic metamaterial rod with spatial periodic distribution and periodical material change, as sketched in Fig. 1. The length of each unit-cell is L. For the spatial periodic distribution the band gaps are generated by Bragg scattering, which appears around frequencies governed by the Bragg condition $L = n(\lambda/2)(n = 1, 2, ...)$, where λ is the unit-cell wavelength.

The longitudinal vibration of the finite rod can be modelled by SEM (Lee, 2009), where the spectral dynamic stiffness element matrix is given by,

$$\mathbf{D} = EA\beta \begin{bmatrix} \cot(\beta L) & -\csc(\beta L) \\ -\csc(\beta L) & \cot(\beta L) \end{bmatrix}, \tag{1}$$

and the spectral dynamic stiffness global matrix is $\mathbf{D}$, provided that the structure geometry is uniform.
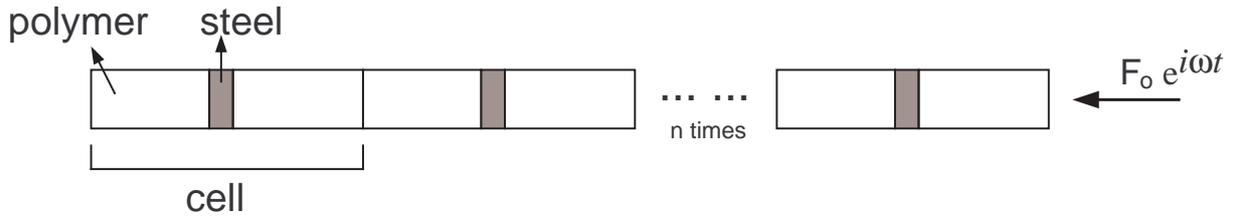

Figura 1: Metamaterial rod with periodic cells of polymer and steel

Rearranging the Eq.(1) as the Transfer Matrix formulation one produces,

$$\underbrace{\left\{ \begin{matrix} \mathbf{u}_r \\ -\mathbf{F}_r \end{matrix} \right\}}_{\mathbf{q}_r} = \underbrace{\begin{bmatrix} -\mathbf{D}_{lr}^{-1}\mathbf{D}_{ll} & -\mathbf{D}_{lr}^{-1} \\ \mathbf{D}_{rl} - \mathbf{D}_{rr}\mathbf{D}_{lr}^{-1}\mathbf{D}_{ll} & -\mathbf{D}_{rr}\mathbf{D}_{lr}^{-1} \end{bmatrix}}_{\mathbf{T}} \underbrace{\left\{ \begin{matrix} \mathbf{u}_l \\ \mathbf{F}_l \end{matrix} \right\}}_{\mathbf{q}_l}. \tag{2}$$

where $\mathbf{T}$ is the transfer matrix that relates the left state vectors $\mathbf{q}_l$ with the right state vector $\mathbf{q}_r$ of the unit-cell. Let's consider now consecutive unit-cells, m and $m + 1$, in the structure shown in Fig. 1. The displacement continuity condition, $\mathbf{u}_r{}^{(m)} = \mathbf{u}_l{}^{(m+1)}$ and the force balance $\mathbf{F}_r{}^{(m)} = -\mathbf{F}_l{}^{(m+1)}$, produces $\mathbf{q}_r{}^{(m)} = \mathbf{q}_l{}^{(m+1)}$. By substituting in the Eq. (2)

$$\mathbf{q}_l{}^{(m+1)} = \mathbf{T}\mathbf{q}_l{}^{(m)}. \tag{3}$$

From this equation the transfer matrix of a cell $\mathbf{T_{cell}}$ can be obtained making:

$$\mathbf{T_{cell}} = \mathbf{T_{poly}T_{steel}T_{poly}} \tag{4}$$

and solving for the whole structure,

$$\mathbf{q}_l{}^{(n)} = [\mathbf{T_{cell}}]^{\mathbf{n}}\mathbf{q}_l{}^{(1)}. \tag{5}$$

where n is the number of substructures or cells.

3. ENERGY FLOW ANALYSIS

One of the main advantages in analysing structural problems using energy and flux variables is the fact that they have a simple physical interpretation. By the use of energy flux equations in structures is possible to identify paths of energy propagation and to make appropriate changes in design to minimize vibrational energy (Noiseux, 1970).

According to Cho and Bernhard (1998), energy differential equation that represents longitudinal wave propagating in a rod is given by,

$$\frac{\partial^2 \langle \bar{e} \rangle}{\partial x^2} - \Psi^2 \langle \bar{e} \rangle = 0, \tag{6}$$

where, $\Psi = \eta \omega / c_g$ and $\langle \bar{e} \rangle$ represents spatial-time average. Frequency of excitation symbol is ω, η is the structural loss factor and c_g is the group velocity given by $c_g = 2 \left(\omega^2 E/A \right)^{1/4}$. Time average for energy flux (q) is related to energy density (e) by the following equation,

$$\langle \bar{e} \rangle = -\frac{c_g^2}{\eta \omega} \frac{d \langle \bar{e} \rangle}{dx} \tag{7}$$

General solution of Eq. (7) is:

$$\langle \bar{e} \rangle = H e^{\Psi x} + G e^{-\Psi x}, \tag{8}$$

where H and G are constants that can be obtained from boundary conditions of energy density and flux.

3.1 ENERGY SPECTRAL ELEMENT FOR A ROD

Energy Spectral Element Method (ESEM) uses the same methodology as Spectral Element Method (SEM) (Doyle, 1997), dynamic stiffness matrices are obtained using exact shape functions, which are established using exact solutions of the energy differential equation (8). Then, they can be used to compute energy in every position along the length of structure. Figure 2 shows a two-noded rod element, with an uniform transversal section, subjected to a energy density and flux on both ends.
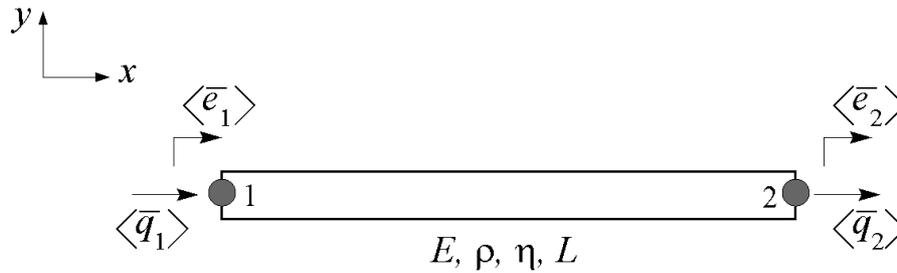

Figura 2: Nodal energy density and flux for a bar energy element.

Applying boundary conditions as shown in Figure 2, one can obtain a dynamic relation between energy density and energy flux, according to the following expression (Santos *et al.*, 2008):

$$\left\{ \begin{array}{c} \langle \bar{q}_1 \rangle \\ \langle \bar{q}_2 \rangle \end{array} \right\} = c_g \left[\begin{array}{cc} \coth \left(L \eta \omega / c_g \right) & -\text{csch} \left(L \eta \omega / c_g \right) \\ -\text{csch} \left(L \eta \omega / c_g \right) & \coth \left(L \eta \omega / c_g \right) \end{array} \right] \left\{ \begin{array}{c} \langle \bar{e}_1 \rangle \\ \langle \bar{e}_2 \rangle \end{array} \right\} \tag{9}$$

$$K_E = \left[\begin{array}{cc} \coth \left(L \eta \omega / c_g \right) & -\text{csch} \left(L \eta \omega / c_g \right) \\ -\text{csch} \left(L \eta \omega / c_g \right) & \coth \left(L \eta \omega / c_g \right) \end{array} \right] \tag{10}$$

where K_E is the dynamic energy flux matrix for a rod element.

3.2 CONNECTION MATRIX

Connection Matrix is used to quantify all that is reflected and transmitted in both sides of the joint between two different materials due to an incident wave (3). In order to evaluate this, coefficients that quantify this energy exchange on the joint are used, they are named coefficient of reflection (r) and transmission (t).

The element connection matrix (C^e) for the longitudinal wave propagation is represented by:

$$\left\{ \begin{array}{c} \langle \bar{q}_1 \rangle \\ \langle \bar{q}_2 \rangle \end{array} \right\} = \frac{t}{2r} \left[\begin{array}{cc} cg_1 & -cg_2 \\ -cg_2 & cg_1 \end{array} \right] \left\{ \begin{array}{c} \langle \bar{e}_1 \rangle \\ \langle \bar{e}_2 \rangle \end{array} \right\} \tag{11}$$

where

$$t = \frac{P_{transmitted}}{P_{injected}}, \quad r = \frac{P_{reflected}}{P_{injected}} \tag{12}$$
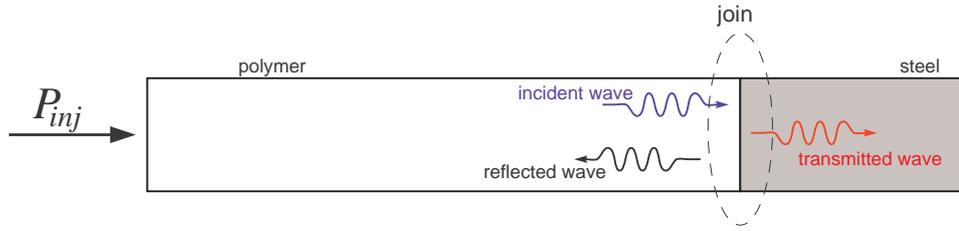
Figura 3: Relation between incident and transmitted waves at the joint.

Combining Eq. (11) and (13) and computing r and t one obtains the following equation system,

$$\left\{ \begin{array}{c} \langle \bar{q}_1 \rangle \\ \langle \bar{q}_2 \rangle \end{array} \right\} = (\mathbf{K}_E + C^e) \left\{ \begin{array}{c} \langle \bar{e}_1 \rangle \\ \langle \bar{e}_2 \rangle \end{array} \right\} \tag{13}$$

4. NUMERICAL RESULTS

Results were obtained for a rod with periodic structure made of steel and a polymeric material with 20 cells. Cross section area is the same for both materials, $A = 0.0125 \times 0.0125 \, \mathrm{m}^2$. Table 1 summarize properties for both materials.

Tabela 1: Properties used for numerical simulation.

Property	Steel	Polymer
Elastic Modulus (E, Pa)	210×10^9	0.48×10^9
Density (ρ, kg/m^3)	7800	935
Damping factor (η)	0.01	0.02

Figure 4 shows the comparison between results for transmittance of SEM and Transfer Matrix. It is possible to identify bandgaps on vibration transmittance. Results show good agreement within this frequency range.
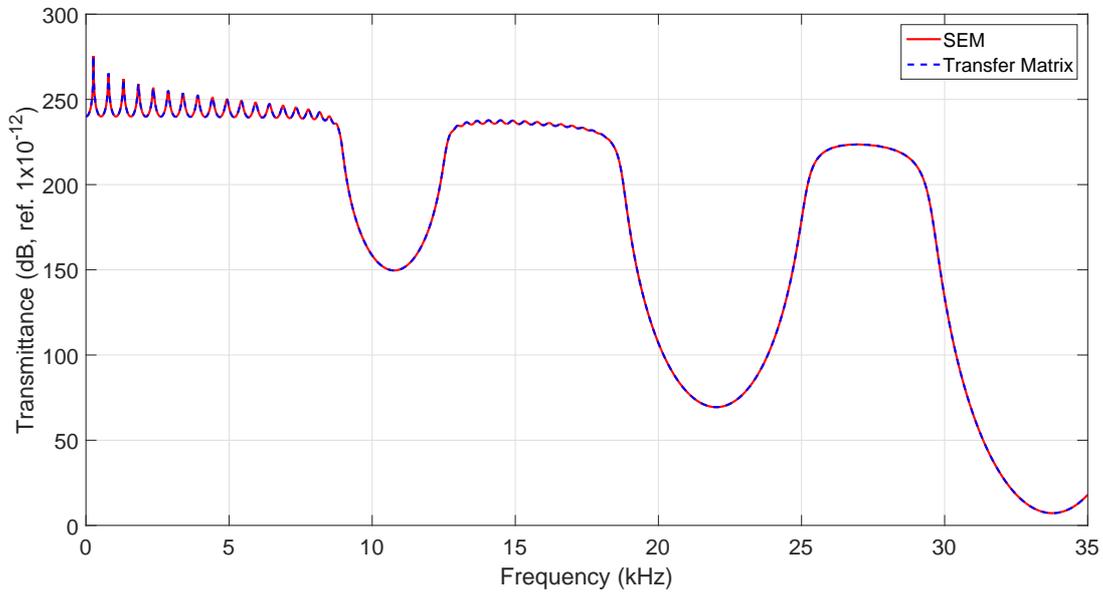

Figura 4: Transmittance Comparison of SEM and Transfer Matrix

In Fig. 5(a) displacement of rod computed at driver-point shows bandgaps as a non oscillating displacement (approximate frequency range: 10-12.5 kHz, 18-25 kHz ...). In a similar manner one can see the same effect in Fig. 5(b), where is computed energy density is presented instead.

5. CONCLUSION

An analysis of a metamaterial was made from the point of view of energy flow using ESEM. Other methods, SEM and Transfer Matrix, were used to validate the results. The rod metamaterial dynamic behaviour was simulated and results indicate that the bandgap phenomenon can also be predicted using energy formulations. As a result, in this rod, waves cannot propagate freely through the periodic structure within some frequency ranges.
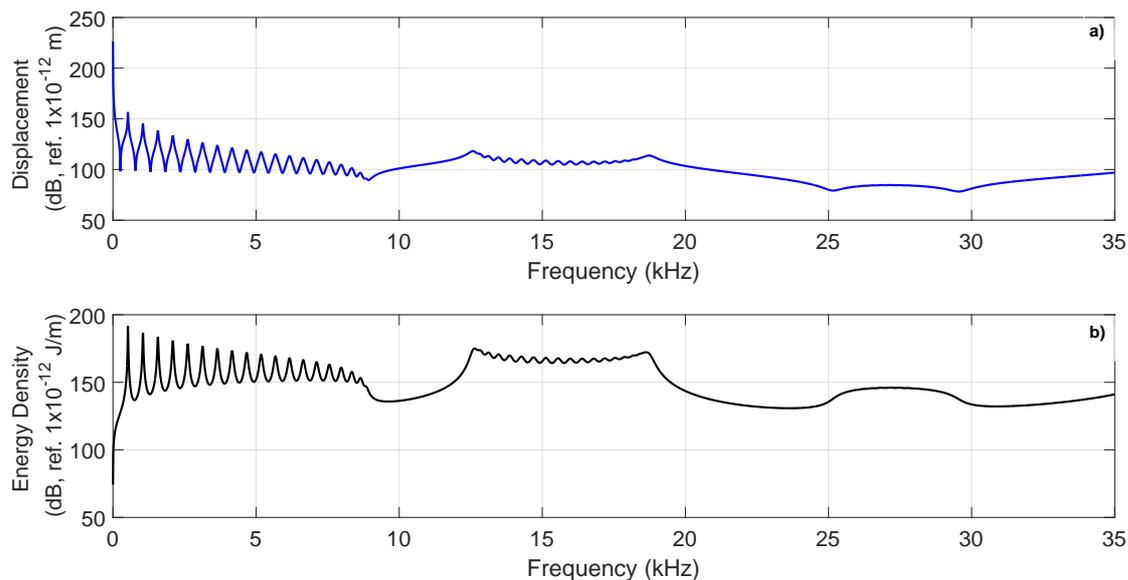
Figura 5: Comparison of displacement and energy density at position $x = 0$.

6. ACKNOWLEDGEMENTS

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8. AUTHORS RESPONSIBILITIES

The authors are the only ones responsible for the contents of this paper.