SDRE CONTROL APPLIED ON A NONLINEAR INVERTED PENDULUM SYSTEM ON A MOVABLE PLATFORM

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Abstract: *This paper presents the LQR and SDRE controls application in the inverted pendulum on a movable platform control. The proposed controls have the objective to control the pendulum angle and the platform position considering the nonlinear system mathematical model. From the results, it was possible to verify that the linear control (LQR) was not able to control the nonlinear system. The SDRE control efficiently controlled the nonlinear system and proved to be robust to the parametric variations.*

Keywords: *Inverted Pendulum, Nonlinear System, LQR, SDRE, Parametric Variations.*

1. INTRODUCTION

The inverted pendulum is one of the mechanical systems commonly used on controller studies. The reason of such system is widely used in this area is linked to the nonlinearity associated with the instability of most of its critical points. This makes the development of controllers to stabilize the system be challenge. (Ogata, 2003; Wang, 2011)

Figure 1 shows the inverted pendulum system on a movable platform.

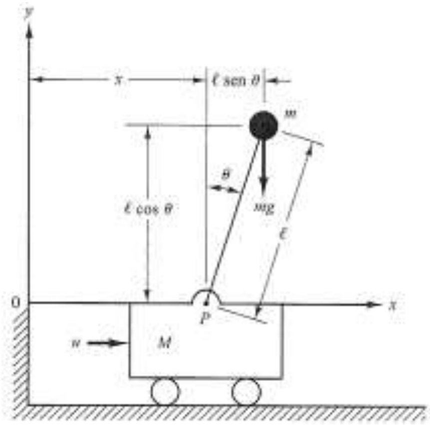

Figure 1. Inverted pendulum system on a movable platform (Ogata, 2003)

This work shows how the system behaves when it uses optimal controllers dependent and non-dependent on the state variables. When they are dependent on the state variables, LQR (Linear Quadratic Regulator) controller is applied due to its dependence on the state variables and it is applicable to linear systems. In the second case, the SDRE (State-Dependent Riccati Equation) controller is applied and it is applicable to nonlinear systems (Dorf and Bishop, 2009; Cloutier, 1997; Ogata, 2003).

In order to verify the controllers performance designed for linear systems when the objective is stabilize a variable in an unstable system point the LQR controller was applied in the nonlinear inverted pendulum to analyze the pendulum angle and the movable platform positioning.

To complement the analysis of the nonlinearly modelled system, the SDRE control was applied to verify the control robustness when there are parametric uncertainties (Tusset *et. al*, 2017).

2. MATHEMATICAL MODELLING

The inverted pendulum is linearly modelled considering pendulum angle oscillation negligible (Dorf and Bishop, 2009; Ogata, 2003). This consideration simplifies the motion equation since the momentum generated by the pendulum oscillation around point P (Fig. 1) tends to zero.

Equations (1) and (2) exhibit the inverted pendulum linearized model represented by Fig. 1.

$$Ml\ddot{\theta} = (M+m)g\theta - u \tag{1}$$
$$M\ddot{x} = u - mg\theta \tag{2}$$

Where: M represents the movable platform mass, m represents the pendulum mass, l represents the pendulum length, g is the gravitational acceleration, u represents the controller, θ = pendulum angle and x = movable platform position.

Equation (2) is rewritten in state-space form resulting in:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ \frac{M+m}{Ml}g & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ -\frac{m}{M}g & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ -\frac{1}{Ml}\\ 0\\ \frac{1}{M}\end{bmatrix}u \tag{3}$$

Where: $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = x$ and $x_4 = \dot{x}$.

The mathematical model of the nonlinear inverted pendulum model, represented in Fig. 1, is given by

$$\ddot{\theta} = -\frac{\mu\sigma}{\lambda}\dot{\theta}^2 \sin\theta\cos\theta - \frac{\eta\sigma}{\lambda}u\cos\theta - \frac{\alpha}{\lambda}\sin\theta \tag{4}$$

$$\ddot{x} = \frac{\mu}{\lambda}\dot{\theta}^2\sin\theta + \frac{\mu\alpha}{\lambda}\sin\theta\cos\theta + \frac{\eta}{\lambda}u \tag{5}$$

Where: M represents the movable platform mass, m represents the pendulum mass, l represents the pendulum length, g is the gravitational acceleration, u represents the controller, θ = pendulum angle, x = movable platform position and the parameters: $\mu = \frac{ml}{(m+M)}$, $\eta = \frac{1}{(m+M)}$, $\alpha = \frac{mgl}{(I+ml^2)}$, $\sigma = \frac{ml}{(I+ml^2)}$ and $\lambda = (1-\sigma\mu\cos^2\theta)$.

Equation (6) represents the nonlinear system motion equation in state-space form.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & -\frac{\mu\sigma}{\lambda}x_2\sin x_1\cos x_1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{\mu}{\lambda}x_2\sin x_1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ -\frac{\eta\sigma}{\lambda}\cos x_1\\ 0\\ \frac{\eta}{\lambda}\end{bmatrix}u \tag{6}$$

Where: $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = x$ and $x_4 = \dot{x}$.

3. QUADRATIC OPTIMAL CONTROL

The quadratic optimal regulator has the objective to minimize the system cost function through state feedback. Equation (7) represents the function for systems with linear behavior.

$$J = \int_0^\infty (x^* Q x + u^* R u) dt \quad (7)$$

Where: Q e R are matrices that determine the state variables relevance.

For the controller development, it is necessary to have the equations of motion written in vector-matrix terms. In other words, the representation must be in state-space notation, as shown in Eq. (8).

$$\dot{x} = Ax(t) + Bu(t) \quad (8)$$

From Eq. (9), it is possible to obtain the control u, as given by

$$u = -Kx(t) = -R^{-1} B^* Pe \quad (9)$$

Where: $\mathbf{e} = \begin{bmatrix} x_1 - x_1^* & x_2 - x_2^* & x_3 - x_3^* & x_4 - x_4^* \end{bmatrix}$ represents the error of the control, x_1^* the desired angle, x_2^* the desired angular velocity, x_3^* the desired platform position, x_4^* the desired platform velocity, and P a symmetric matrix obtained by the *Riccati* Eq. (10).

$$PA + A^T P - PBR^{-1} B^T P + Q = 0 \quad (10)$$

This way to minimize the cost function can also be used in systems that behave nonlinearly. The strategy implemented by Cloutier (1997) consists on considering matrices A and B of Eq. (8) dependent on state variables. Rewriting the equation, this results in the Eq. (11).

$$\dot{x} = A(x)x(t) + B(x)u(t) \quad (11)$$

By introducing this concept into the *Riccati* equation, the matrix P can be obtained from Eq. (12).

$$A^T(x)P + PA(x) - PB(x)R^{-1}(x)B^T(x)P + Q(x) = 0 \quad (12)$$

The system feedback occurs from Eq. (13).

$$u = -K(x)x(t) = -R^{-1}(x)B^*(x)P(x)e \quad (13)$$

Equation (14) represents the performance index function to minimize costs considering state variables dependence.

$$J = \int_0^\infty (x^T Q(x) x + u^T R(x) u dt \quad (14)$$

4. ANALYSIS OF PARAMETRIC SENSITIVITY

Mathematical models do not express all the inaccuracies inherent in the real physical system. In order to ensure that the systems perform their function even with the uncertainties associated with it, robust systems are proposed. (Dorf and Bishop, 2009)

This study considers uncertainties in the following system construction parameters: platform mass, pendulum mass and pendulum length. Uncertainty may be associated with n parameters variations, which may result in z different system behaviors. Tusset *et al*. (2017) proposes a method that systematizes the variations and covers a wide range of behaviors.

The method consists of randomly varying system construction parameters in different percentage ranges. Eq. (15) represents this reasoning mathematically.

$$f = (1-h)p + (2h)pr \quad (15)$$

Where: h represents the variation parameter percentage, p represents the nominal parameter value and r represents random value.

5. NUMERICAL SIMULATION

In order to verify the performance of the tested controllers, the movable platform must move from position 0 m to position 5 m and remain on such position when the platform reaches there. Throughout the translation, the pendulum must remain vertically upwards and remain in this position when the movable platform moves until the intended position.

During the numerical simulations, the following parameters were considered: $l = 0.2$[m], $m = 0.12$[kg], $M =$ 12[kg], $g = 9.81$[m/s^2].

The initial positions of the state variables started at zero (pendulum angle, angular velocity, movable platform position and movable platform velocity). Such initial positions are represented in mathematical terms, such as: $x_1(0) =$ 0, $x_2(0) =$ 0, $x_3(0) =$ 0 e $x_4(0) =$ 0.

Representing the final states in mathematical terms, the desired pendulum angle is $x_1^* = 0$, the desired angle velocity is $x_2^* = 0$, the desired movable platform position is $x_3^* = 5$, and the desired movable platform velocity is $x_4^* = 0$.

In order to generate the gain matrix K (Eq. (9) and Eq. (13)) the Matlab® function *lqr* (A,B,Q,R) was used, where

$$Q = 100\begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ a } R = 0.1 .$$

In order to analyze the SDRE controller robustness the control was subjected to random variations of ± 5%, ± 10%, ± 15% and ± 20% in the parameters. Four situations were considered. First, the length of the pendulum, the mass of the pendulum and the mass of the movable platform were kept separately, while the other parameters were maintained without any parametric variation, and concluding with the simultaneous variation of the three parameters.

Using the linear controller (LQR), suitable for linear systems, in the nonlinear mathematical model, Figs. 2 were obtained.

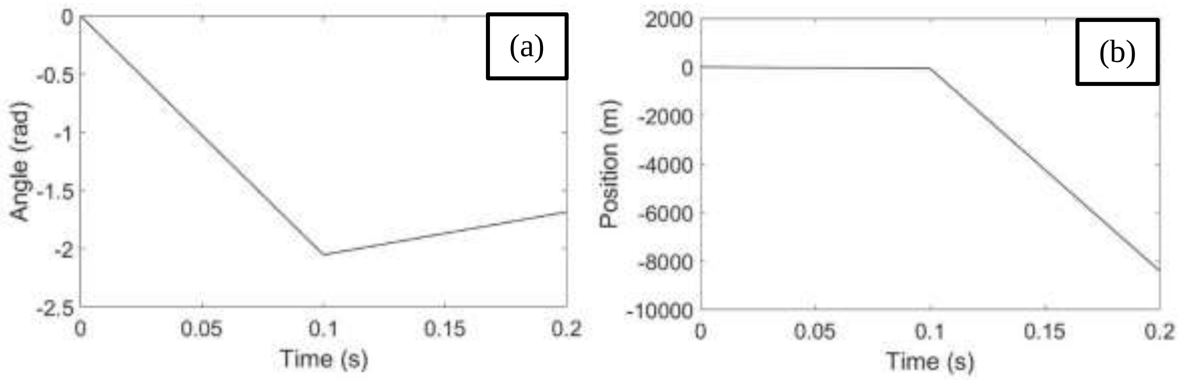

Figure 2. Application of the LQR control in the non-linear system. (a) Position of the pendulum angle with respect to time. (b) Position of the mobile platform in relation to time.

By the results, it is possible to verify that the LQR control do not control the studied nonlinear model. This became evident since the angle of the pendulum in 0.2s varied more than one quadrant. In an attempt to return to the system stabilization with the inverted pendulum positioned vertically upwards, the platform covered approximately 8km.

Applying the SDRE control, which is more appropriate for nonlinear systems, in the nonlinear mathematical model, Figs. 3 were obtained.

As can be observed the control SDRE controlled the nonlinear system efficiently. The pendulum angle had variations close to 1.5 degrees in both directions, however after 20s it stabilized upright. With this stabilization, the movable platform moved from 0m to 5m efficiently and remained in the final position with the pendulum stabilized at 0 degrees.

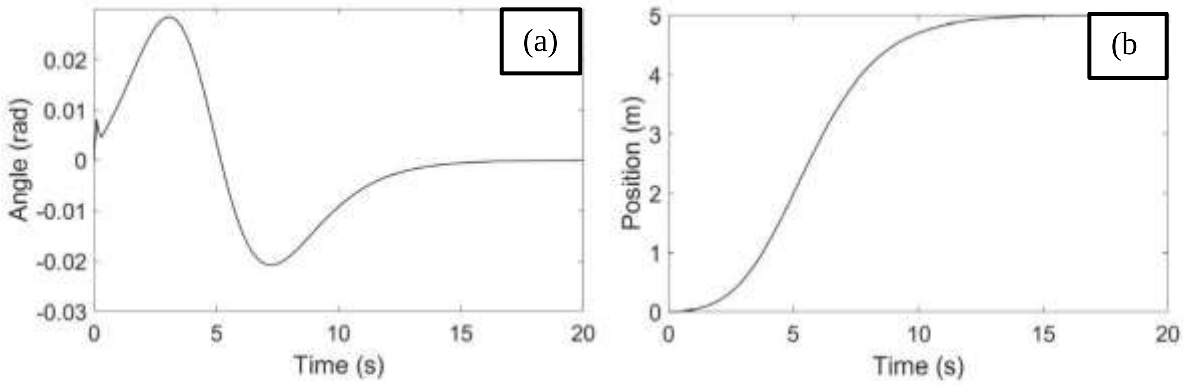

Figure 3. Application of the SDRE control in the non-linear system. (a) Position of the pendulum angle with respect to time. (b) Position of the mobile platform in relation to time.

With the confirmation that the SDRE controller controlled the nonlinear system, it was possible to verify the system robustness. The graph was generated considering the pendulum position and the movable platform response for variations of the physical parameters (pendulum length, pendulum mass, platform mass and the three varying simultaneously) in ± 5% ± 10% ± 15% and ± 20%. This response was compared to the performance without the parametric variations and, from it, the parametric error graphs were generated.

In Figs. 4 is possible to observe the error carried out by the control considering the angular position when the parameters varied in ± 15% of its nominal value.

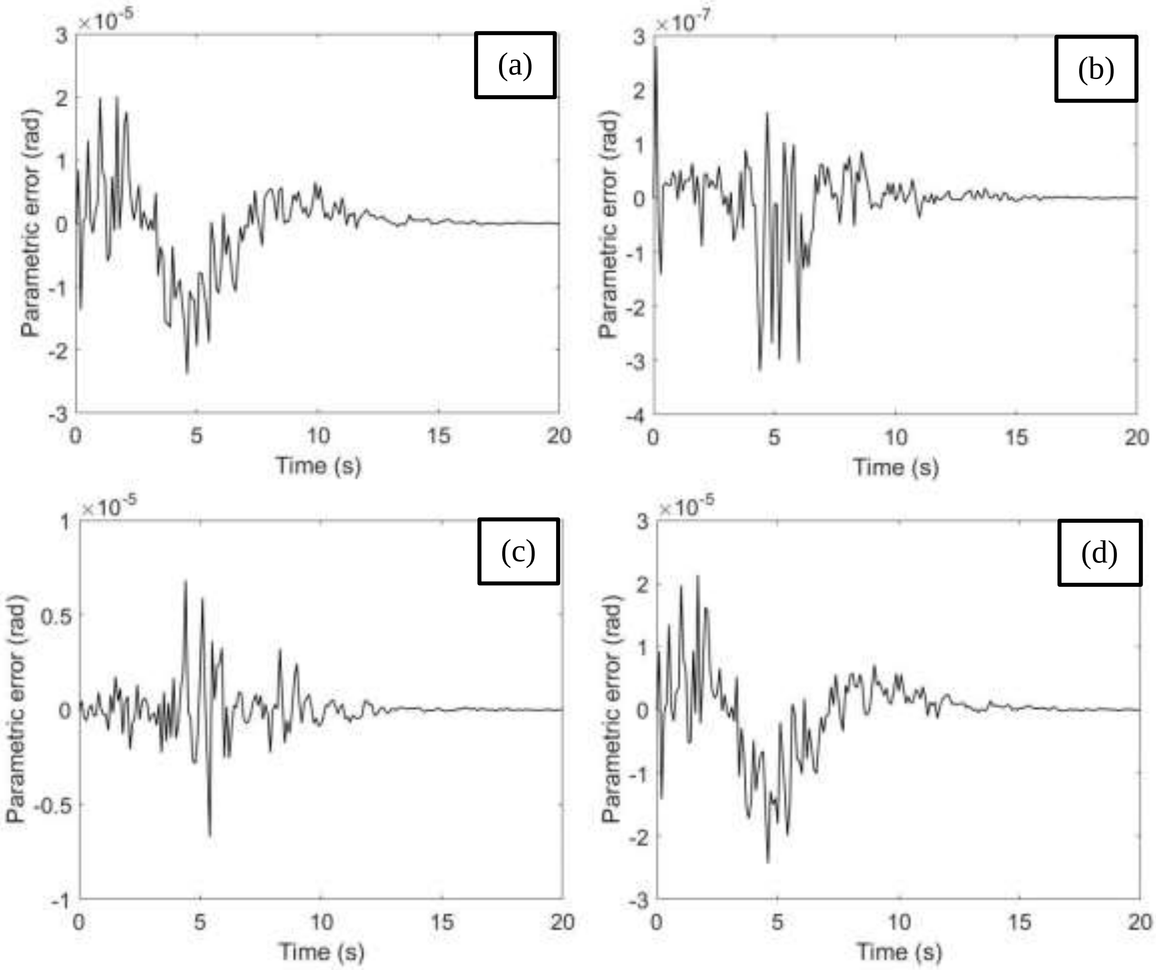

Figure 4. Parametric error response that occurs in the pendulum angle response. (a) variation applied only to the pendulum length. (b) variation applied only to the pendulum mass. (c) variation applied only to the platform mass. (d) variation applied to all parameters simultaneously.

In Fig. 5 it is possible to observe the error carried out by the control considering the movable platform position when the parameters vary in ± 15% of its nominal value.

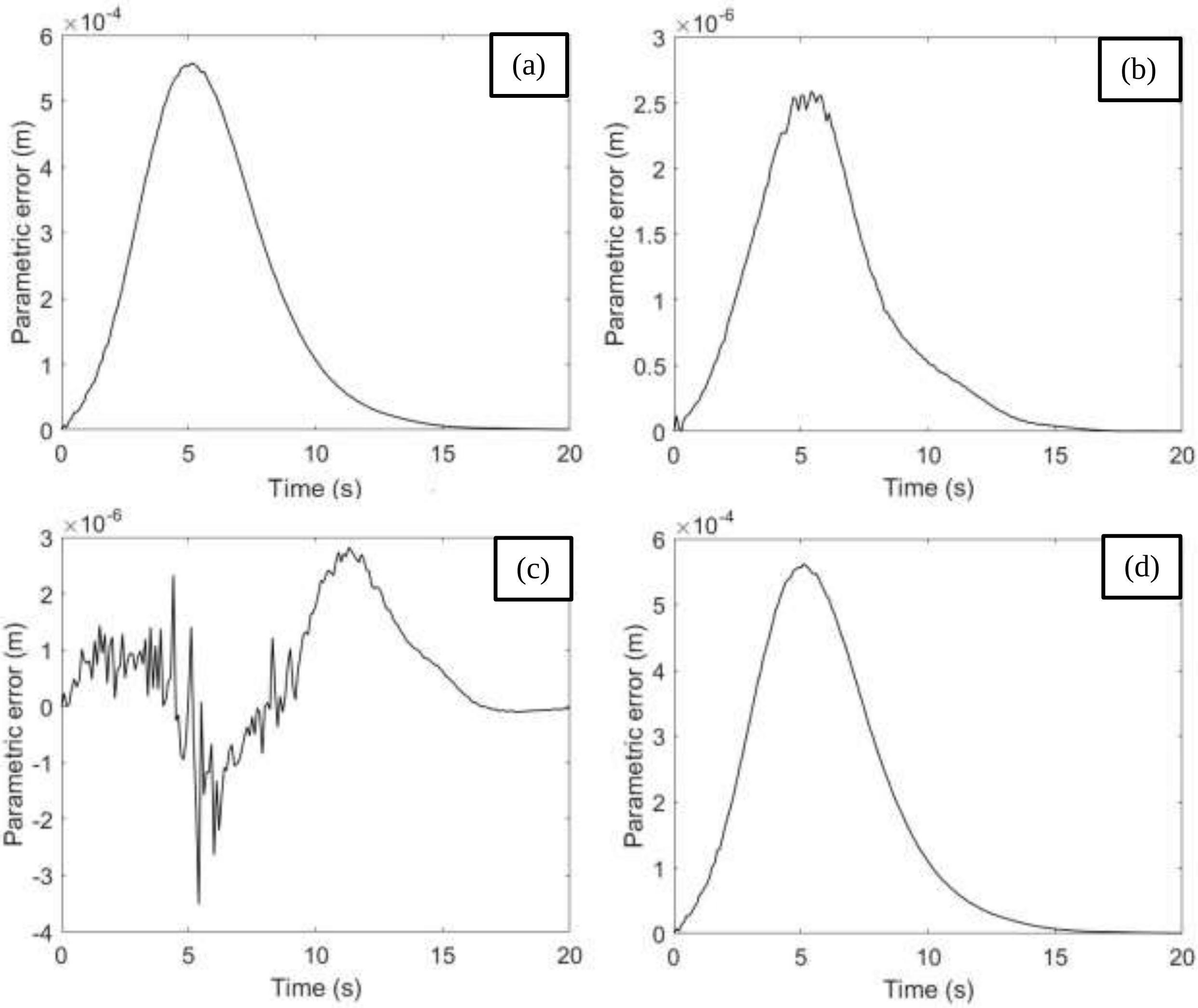

Figure 5. Parametric error response that occurs in the movable platform position response. (a) Variation applied only to the pendulum length. (b) Variation applied only to the pendulum mass. (c) Variation applied only to the platform mass. (d) Variation applied to all parameters simultaneously.

It is possible to observe in Fig. 4 and Fig. 5 that the SDRE controller proved to be robust for parametric variations, since it performed its function despite of the adverse situation in which it was exposed.

Each parametric variation range generated the parametric error graph associated with the pendulum angle and the movable platform position for each exposed situation. Table 1 represents these compiled data to show the largest approximate module amplitude present in each graph.

Table 1. Parametric error associated to pendulum angle and movable platform position in different parametric variation ranges

	±5%		±10%		±15%		±20%	
Parameter	Pendulum angle (rad)	Movable platform position (m)	Pendulum angle (rad)	Movable platform position (m)	Pendulum angle (rad)	Movable platform position (m)	Pendulum angle (rad)	Movable platform position (m)
	Error	Error	Error	Error	Error	Error	Error	Error
L	2.5 10^{-5}	9 10^{-5}	9 10^{-5}	1.1 10^{-3}	2.5 10^{-5}	5.5 10^{-4}	2 10^{-5}	14 10^{-5}
m	1.8 10^{-7}	7 10^{-7}	3 10^{-7}	4 10^{-7}	3 10^{-7}	2.5 10^{-6}	4.5 10^{-7}	2.7 10^{-6}
M	4 10^{-6}	13 10^{-6}	6 10^{-6}	4.5 10^{-5}	7 10^{-6}	3.5 10^{-6}	7 10^{-6}	17 10^{-6}
l m M	2.5 10^{-5}	8 10^{-5}	9 10^{-5}	1 10^{-3}	2.5 10^{-5}	5.5 10^{-4}	2 10^{-5}	12 10^{-5}

It is noted that the pendulum length is, in all the analyzed ranges, the parameter that most interfered in the system response submitted to the controller when parametric variations occur. The interference is realized in the pendulum angle and in the movable platform position.

6. CONCLUSIONS

From the analysis obtained through the numerical simulation results, it is possible to observe that to control the nonlinear inverted pendulum system with the controller designed for linear systems is not suitable for the studied case.

Regarding the SDRE control sensitivity to the parametric uncertainties, the numerical results showed that the SDRE is robust even when the parameter: pendulum length, pendulum mass and platform mass vary simultaneously and randomly up to ± 20%.

7. ACKNOWLEDGMENTS

The authors acknowledge support by CNPq, CAPES, FAPESP and FA, all Brazilian research funding agencies.

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9. RESPONSIBILITY NOTICE

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