A NODAL METHOD FOR SOLVING THE TWO-DIMENSIONAL PARTICLE TRANSPORT EQUATION

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Abstract: *In this work, a two-dimensional neutron transport problem defined in a homogeneous medium is treated. The domain configuration is defined by a fixed source enclosed in a rectangular region surrounded by vacuum boundary conditions. A combination of nodal schemes with explicit solutions for the transverse-integrated equations, using the Analytical Discrete Ordinates (ADO) Method, is used to evaluate averaged scalar fluxes for regions of the domain. The unknown angular fluxes on the contours, that appear from the derivation of the nodal schemes, are added to the source term and differently of previous works, here those unknowns are approximated by three types of functions: constant, linear, and exponential. In addition, an alternative multidimensional quadrature scheme, named QR, was used to describe the discrete ordinates directions. Such scheme allows the use of higher-order approximations than the classic LQN (level symmetric) quadrature scheme. Numerical results obtained by the methodology, which may be used in radiative transfer applications, are presented and compared with the literature. In particular, the effects of the boundary fluxes approximations are analyzed.*
Keywords: *Two dimensional transport , Discretes ordinates, Fixed-source problems, Homogenous medium*

1. INTRODUCTION

Nodal schemes have been widely used for solving multidimensional transport problems related to several applications like "*oil well logging*" (Badruzzaman, 1985; Azmy, 1988); neutron and radiation transport (Lawrence, 1986; Barros and Larsen, 1992; Rohde *et al.*, 2012; Okumura *et al.*, 2014; Naqa *et al.*, 2003). Due to the inherent integration procedure the complexity of the original model is reduced and, in consequence, such approaches are amenable to analytical techniques. Nodal methods are also recognized for its great performance on coarse meshes. In this context, we use the ADO method (Analytical Discrete ordinates) (Barichello and Siewert, 1999) along with nodal schemes to solve the discrete ordinates form of the two-dimensional neutron transport equation in Cartesian geometry. In contrast to the majority of numerical methods, such approach has provided explicit expressions for the averaged angular fluxes in terms of the spatial variables, Form fixed source problems have been considered (Barichello *et al.*, 2011; Prolo Filho and Barichello, 2014; Tres *et al.*, 2014; Picoloto *et al.*, 2015).

A relevant issue related to nodal methods is the definition of auxiliary equations needed to represent unknown angular fluxes on the contours of the domain or interfaces, arising from the integration procedure. In order to bring greater contributions to the solution of homogeneous problems using ADO method, in this paper three different approaches are proposed to the treatment of such unknowns: constant, linear and exponential approximations. In addition, the discrete angular directions are represented trough the use of different sets of numerical quadrature schemes. The LQ_N (Lewis and Miller, 1984) (Level Symmetric Quadrature) and QR (Abu-Shumays, 1977) (Quadruple Range). The latter allow us to consider higher order number of directions than the classical scheme. The final goal is to perform a general analysis in order to establish reference results.

2. FORMULATION

We begin with the discrete ordinates approximation of the time-independent two dimensional neutron transport equation, in a non-multiplicative medium, with isotropic scattering and one energy group, in a rectangular domain D, with $x \in [0, a]$ and $y \in [0, b]$, written in discrete ordinates form as (Lewis and Miller, 1984)

$$\mu_i \frac{\partial}{\partial x}\Psi(x, y, \mathbf{\Omega}_i) + \eta_i \frac{\partial}{\partial y}\Psi(x, y, \mathbf{\Omega}_i) + \sigma_t \Psi(x, y, \mathbf{\Omega}_i) = Q(x, y) + \sigma_s \sum_{k=1}^{M} w_k \Psi(x, y, \mathbf{\Omega}_k) \tag{1}$$

for $i = 1, \ldots, M$, where M it is defined according to the quadrature used; w_i are the weights associated to the directions $\mathbf{\Omega}_i = (\mu_i, \eta_i)$; $Q(x, y)$ is the isotropic neutron source term; and σ_t and σ_s are the total and scattering macroscopic cross sections, respectively.

We consider the domain subdivided into regions named as r defined as $x \in [a_{m-1}, a_m]$ and $y \in [b_{m-1}, b_m]$, with $0 \le a_{m-1} < a_m \le a$ and $0 \le b_{m-1} < b_m \le b$. Here m indicates the number of divisions of each spatial variables interval of definition, as represented in Figure 1. Following (Barichello *et al.*, 2011), we establish an ordering on the directions $\mathbf{\Omega}_i$ such that for indices $i = 1, \ldots, M/2$ the directions have coordinate $\mu_i > 0$; and $\mu_i < 0$ for indices $i = M/2 + 1, \ldots, M$.

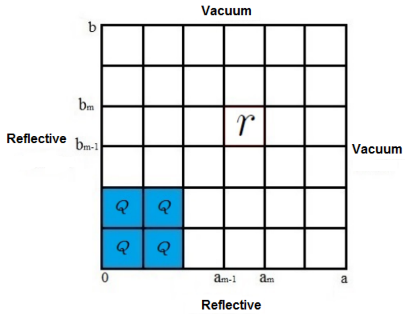

Figura 1: **Representation of the domain.**

Therefore, using nodal method techniques to obtain the the one dimensional transverse integrated equations in x direction on a region r, we integrate Equation (1) for all $y \in [b_{m-1}, b_m]$, obtaining

$$\mu_i \frac{d}{dx}\Psi_{yr}(x,\mathbf{\Omega}_i) + \sigma_{tr}\Psi_{yr}(x,\mathbf{\Omega}_i) = Q_{yr}(x,\Omega_i) + \sigma_{sr}\sum_{k=1}^{M/2} w_k\left[\Psi_{yr}(x,\mathbf{\Omega}_k) + \Psi_{yr}(x,\mathbf{\Omega}_{k+M/2})\right], \tag{2}$$

$$-\mu_i \frac{d}{dx}\Psi_{yr}(x,\mathbf{\Omega}_{i+M/2}) + \sigma_{tr}\Psi_{yr}(x,\mathbf{\Omega}_{i+M/2}) = Q_{yr}(x,\Omega_{i+M/2}) + \sigma_{sr}\sum_{k=1}^{M/2} w_k\left[\Psi_{yr}(x,\mathbf{\Omega}_k) + \Psi_{yr}(x,\mathbf{\Omega}_{k+M/2})\right], \tag{3}$$

for $i = 1, \ldots, M/2$, where we defined the integrated angular flux for x-direction

$$\Psi_{yr}(x,\mathbf{\Omega}_i) = \frac{1}{\alpha_m}\int_{b_{m-1}}^{b_m} \Psi_r(x,y,\mathbf{\Omega}_i)dy, \tag{4}$$

and source terms as

$$Q_{yr}(x,\mathbf{\Omega}_i) = Q_{yr}(x) - \frac{\eta_i}{\alpha_m}\left[\Psi_r(x,b_r,\mathbf{\Omega}_i) - \Psi_r(x,b_{r-1},\mathbf{\Omega}_i)\right], \tag{5}$$

$$Q_{yr}(x) = \frac{1}{\alpha_m}\int_{b_{m-1}}^{b_m} Q_r(x,y)dy, \tag{6}$$

where $\alpha_m = b_m - b_{m-1}$ and $i = 1, \ldots, M$.

We proceed similarly to obtain such a system in terms of the variable y. Again, following previous works (Barichello *et al.*, 2011), we associate indices $i = 1, \ldots, M/2$ to directions with coordinate $\eta_i > 0$ and indices $i = M/2+1, \ldots, M$ to directions with coordinates $\eta_i < 0$. Integrating Equation (1) for every $x \in [a_{m-1}, a_m]$, we obtain

$$\eta_i \frac{d}{dy}\Psi_{xr}(y,\mathbf{\Omega}_i) + \sigma_{tr}\Psi_{yr}(y,\mathbf{\Omega}_i) = Q_{xr}(y,\Omega_i) + \sigma_{sr}\sum_{k=1}^{M/2} w_k\left[\Psi_{xr}(y,\mathbf{\Omega}_k) + \Psi_{xr}(y,\mathbf{\Omega}_{k+M/2})\right], \tag{7}$$

$$-\eta_i \frac{d}{dy}\Psi_{xr}(y,\mathbf{\Omega}_{i+M/2}) + \sigma_{tr}\Psi_{xr}(y,\mathbf{\Omega}_{i+M/2}) = Q_{xr}(y,\Omega_{i+M/2}) + \sigma_{sr}\sum_{k=1}^{M/2} w_k\left[\Psi_{xr}(y,\mathbf{\Omega}_k) + \Psi_{xr}(y,\mathbf{\Omega}_{k+M/2})\right], \tag{8}$$

for $i = 1, \ldots, M/2$. In these equations, we define the integrated angular flux for y-direction

$$\Psi_{xr}(y,\mathbf{\Omega}_i) = \frac{1}{\beta_m}\int_{a_{m-1}}^{a_m} \Psi_r(x,y,\mathbf{\Omega}_i)dx, \tag{9}$$

and the source terms

$$Q_{xr}(y,\mathbf{\Omega}_i) = Q_{xr}(y) - \frac{\mu_i}{\beta_m}\left[\Psi_r(a_r,y,\mathbf{\Omega}_i) - \Psi_r(a_{r-1},y,\mathbf{\Omega}_i)\right], \tag{10}$$

$$Q_{xr}(y) = \frac{1}{\beta_m}\int_{a_{m-1}}^{a_m} Q_r(x,y)dx, \tag{11}$$

where $i = 1, \ldots, M$ and $\beta_m = a_m - a_{m-1}$.

We note that in Equations (6) and (11), the angular fluxes on the contours of each region, resulting from the integration of Equation (1), were incorporated to the source term. Some of these terms can be obtained from the known boundary conditions of the problem. For the unknown variables as exiting angular fluxes, auxiliary equations have to be introduced; what is usual in nodal schemes. In what follows, we propose different approximations for those unknowns which are incorporated to the non-homogeneous source term. Such approximations as well as the boundary conditions have to be defined in order to derive particular solutions as will be discussed later on.

2.1 Solution by the ADO method in a medium *r*

The transverse integrated one-dimensional equations defined in the previous section, are resolved in this section using ADO method. Thus, for a region r, we propose solutions of the homogeneous problem, for $i = 1, ..., M$, as

$$\Psi_{yr}^H(x, \mathbf{\Omega}_i) = \Phi_{yr}(\nu_r, \mathbf{\Omega}_i) e^{-x/\nu_r}. \tag{12}$$

Substituting Equation (12) in Equations (2) and (3), we obtain

$$-\frac{\mu_i}{\nu_r}\Phi_{yr}(\nu_r, \mathbf{\Omega}_i) + \sigma_{tr}\Phi_{yr}(\nu_r, \mathbf{\Omega}_i) = \sigma_{sr}\sum_{k=1}^{M/2} w_k \left[\Phi_{yr}(\nu_r, \mathbf{\Omega}_k) + \Phi_{yr}(\nu_r, \mathbf{\Omega}_{k+M/2})\right], \tag{13}$$

$$\frac{\mu_i}{\nu_r}\Phi_{yr}(\nu_r, \mathbf{\Omega}_{i+M/2}) + \sigma_{tr}\Phi_{yr}(\nu_r, \mathbf{\Omega}_{i+M/2}) = \sigma_{sr}\sum_{k=1}^{M/2} w_k \left[\Phi_{yr}(\nu_r, \mathbf{\Omega}_k) + \Phi_{yr}(\nu_r, \mathbf{\Omega}_{k+M/2})\right], \tag{14}$$

for $i = 1, ..., M/2$. We obtain, manipulating the equations above and from the definitions

$$\mathbf{U}_{yr}(\nu_r, \mathbf{\Omega}_i) = \Phi_{yr}(\nu_r, \mathbf{\Omega}_i) + \Phi_{yr}(\nu_r, \mathbf{\Omega}_{i+M/2}) \tag{15}$$

and

$$\mathbf{V}_{yr}(\nu_r, \mathbf{\Omega}_i) = \Phi_{yr}(\nu_r, \mathbf{\Omega}_i) - \Phi_{yr}(\nu_r, \mathbf{\Omega}_{i+M/2}) \tag{16}$$

the eigenvalue problem

$$\left[\mathbf{D}_{yr} - \mathbf{A}_{yr}\right]\mathbf{U}_{yr} = \lambda_{yr}\mathbf{U}_{yr}, \tag{17}$$

with

$$\lambda_{yr} = \frac{1}{\nu_r^2}, \tag{18}$$

where $\mathbf{D}_{yr}$ and $\mathbf{A}_{yr}$ are $M/2 \times M/2$ matrices defined as

$$\mathbf{D}_{yr} = diag\left[\left[\frac{\sigma_{tr}}{\mu_1}\right]^2, ..., \left[\frac{\sigma_{tr}}{\mu_{M/2}}\right]^2\right] \tag{19}$$

and the elements of $\mathbf{A}_{yr}$ are given by

$$\mathbf{A}_{yr}(i,j) = \frac{\sigma_{sr}\sigma_{tr}w_j}{2\mu_i^2}, \tag{20}$$

for $i = 1, ..., M/2$ and $j = 1, ..., M/2$. From the solution of the eigenvalue problem, Equation (17), we obtain λ_{jr}, U_{jr} for $j = 1, ..., M/2$, the separation constants follows from Equation(18). Knowing that the separation constants occur in pair, the homogeneous solution on a region r may be written as

$$\Psi_{yr}^H(x, \mathbf{\Omega}_i) = \sum_{j=1}^{M/2}\left[A_{j,r}\Phi_{yr}(\nu_{jr}, \mathbf{\Omega}_i)e^{-(x-a_{m-1})/\nu_{jr}} + A_{j+M/2,r}\Phi_{yr}(\nu_{jr}, \mathbf{\Omega}_{i+M/2})e^{-(a_m-x)/\nu_{jr}}\right], \tag{21}$$

$$\Psi_{yr}^H(x, \mathbf{\Omega}_{i+M/2}) = \sum_{j=1}^{M/2}\left[A_{j,r}\Phi_{yr}(\nu_{jr}, \mathbf{\Omega}_{i+M/2})e^{-(x-a_{r-1})/\nu_{jr}} + A_{j+M/2,r}\Phi_{yr}(\nu_{jr}, \mathbf{\Omega}_i)e^{-(a_r-x)/\nu_{jr}}\right] \tag{22}$$

for $i = 1, ..., M/2$, $x \in [a_{m-1}, a_m]$, where $A_{j,r}$ and $A_{j+M/2,r}$ represent the coefficients of the solution relative to region r.

We proceed similarly to obtain the eigenvalue problem,separation constants and the solution in the y-direction in the same region:

$$\Psi_{xr}^H(y, \mathbf{\Omega}_i) = \sum_{j=1}^{M/2}\left[B_{j,r}\Phi_{xr}(\gamma_{jr}, \mathbf{\Omega}_i)e^{-(y-b_{m-1})/\gamma_{jr}} + B_{j+M/2,r}\Phi_{xr}(\gamma_{jr}, \mathbf{\Omega}_{i+M/2})e^{-(b_m-y)/\gamma_{jr}}\right], \tag{23}$$

$$\Psi_{xr}^H(y,\mathbf{\Omega}_{i+M/2}) = \sum_{j=1}^{M/2}\left[B_{j,r}\Phi_{xr}(\gamma_{jr},\mathbf{\Omega}_{i+M/2})e^{-(y-b_{r-1})/\gamma_{jr}} + B_{j+M/2,r}\Phi_{xr}(\gamma_{jr},\mathbf{\Omega}_i)e^{-(b_r-y)/\gamma_{jr}}\right], \quad (24)$$

for $i = 1,...,M/2$, $y \in [b_{m-1}, b_m]$. Here $B_{j,r}$ and $B_{j+M/2,r}$ represent the coefficients of the solution relative the region and the separation constants γ_{jr} are obtained from the solution of the following eigenvalue problem, similar to Equation (17),

$$[\mathbf{D}_{xr} - \mathbf{A}_{xr}]\,\mathbf{U}_{xr} = \lambda_{xr}\mathbf{U}_{xr} \quad (25)$$

with $\lambda_{xr} = 1/\gamma_r^2$. The matrices $\mathbf{D}_{xr}$ and $\mathbf{A}_{xr}$ of order $M/2$, are defined as

$$\mathbf{D}_{xr} = diag\left[\left[\frac{\sigma_{tr}}{\eta_1}\right]^2,...,\left[\frac{\sigma_{tr}}{\eta_{M/2}}\right]^2\right] \quad (26)$$

$$\mathbf{A}_{xr}(i,j) = \frac{\sigma_{sr}\sigma_{tr}w_j}{2\eta_i^2}, \quad (27)$$

for $i = 1,...,M/2$, $j = 1,...,M/2$ and $r = 1,\dots,R$. It is important to note that from a set M directions, we come to a eigenvalue problem $M/2$.

2.2 General Solution

As in Equations (2), (3), (7) and (8) appears the inhomogeneous source term, one needs to define a particular solution to the problem. Moreover, these sources terms depend on the angular fluxes on the contours which are not known in all directions. Thus we need to define auxiliary equations.

Differently of previous works, we will use three different approaches to approximate the unknown flows in the contours: approaches by constant, linear and exponential approximations. For this, consider the following equations

$$\Psi_r(x,b_m,\mathbf{\Omega}_i) = \varphi_1 D_{m,r,i} \quad (28a)$$
$$\Psi_r(x,b_{m-1},\mathbf{\Omega}_i) = \varphi_1 D_{m-1,r,i}, \quad (28b)$$
$$\Psi_r(a_m,y,\mathbf{\Omega}_i) = \varphi_2 C_{m,r,i}, \quad (28c)$$
$$\Psi_r(a_{m-1},y,\mathbf{\Omega}_i) = \varphi_2 C_{m-1,r,i}, \quad (28d)$$

such that, for constant approximation let us consider $\varphi_1 = \varphi_2 = 1$, linear $\varphi_1 = x$ and $\varphi_2 = y$ and exponential $\varphi_1 = e^{-(a_m-x)/\nu_{r,max}}$ and $\varphi_2 = e^{-(b_m-y)/\gamma_{r,max}}$. With $i = 1,...,M$ and $r = 1,...,R$, where1 m indicating the number of divisions of the domain, $\gamma_{r,max}$ and $\nu_{r,max}$ are the separation constants of each region obtained from the eigenvalue problem solution.

At this point, we defined the presence of a fixed isotropic source of neutrons in Region I,

$$Q(x,y) = \begin{cases} 1, & \text{to } x \in [0,a_1] \text{ e } y \in [0,b_1] \\ 0, & \text{otherwise,} \end{cases} \quad (29)$$

in the integrate form

$$Q_{yr}(x) = Q_{xr}(y) = \begin{cases} 1, & \text{to } r = 1 \\ 0, & \text{elsewhere.} \end{cases} \quad (30)$$

From Equation (30) and Equations (28), we can rewrite the term source of integrated problems y and x, Equations (5) and (10), respectively, the form

$$Q_{yr}(x,\mathbf{\Omega}_i) = \begin{cases} 1 - \dfrac{\eta_i}{b_m - b_{m-1}}\left[D_{m,r,i} - D_{m-1,r,i}\right]\varphi_1, & \text{to } r = 1 \\ \\ -\dfrac{\eta_i}{b_m - b_{m-1}}\left[D_{m,r,i} - D_{m-1,r,i}\right]\varphi_1, & \text{elsewhere} \end{cases} \quad (31)$$

and

$$Q_{xr}(y,\mathbf{\Omega}_i) = \begin{cases} 1 - \dfrac{\mu_i}{a_m - a_{m-1}}\left[C_{m,r,i} - C_{m-1,r,i}\right]\varphi_2, & \text{to } r = 1 \\ \\ -\dfrac{\mu_i}{a_m - a_{m-1}}\left[C_{m,r,i} - C_{m-1,r,i}\right]\varphi_2, & \text{elsewhere,} \end{cases} \quad (32)$$

where $i = 1,\dots,M$, $r = 1,\dots,R$. Since we have defined the terms of the sources integrated problem in y and x, the next step is the deduction of the particular solution to be used.

2.2.1 Particular Solution

In this work, we consider Green's functions to derive following the same development presented in (Barichello *et al.*, 2000; Prolo Filho, 2011). In this way, we define the particular solution for integrated problem y with

$$\Psi^p_{yr}(x,\mathbf{\Omega}_i)=\sum_{j=1}^{M/2}\{A_{j,r}(x)\Phi_{yr}(\nu_{jr},\mathbf{\Omega}_i)+A_{j+M/2,r}(x)\Phi_{yr}(\nu_{jr},\mathbf{\Omega}_{i+M/2})\}, \tag{33}$$

$$\Psi^p_{yr}(x,\mathbf{\Omega}_{i+M/2})=\sum_{j=1}^{M/2}\{A_{j,r}(x)\Phi_{yr}(\nu_{jr},\mathbf{\Omega}_{i+M/2})+A_{j+M/2,r}(x)\Phi_{yr}(\nu_{jr},\mathbf{\Omega}_i)\}, \tag{34}$$

where

$$A_{j,r}(x)=\int_{a_{m-1}}^{x}\left\{\sum_{\alpha=1}^{M}Q_{yr}(\tau,\mathbf{\Omega}_\alpha)A_{j,r,\alpha}\right\}e^{\frac{-(x-\tau)}{\nu_{jr}}}\,d\tau, \tag{35}$$

$$A_{j+M/2,r}(x)=-\int_{x}^{a_m}\left\{\sum_{\alpha=1}^{M}Q_{yr}(\tau,\mathbf{\Omega}_\alpha)A_{j+M/2,r,\alpha}\right\}e^{\frac{-(\tau-x)}{\nu_{jr}}}\,d\tau, \tag{36}$$

with $i=1,...,M/2$, $j=1,...,M/2$, $r=1,...,R$ and $\alpha=1,...,M$, and $A_{j,r,\alpha}$, $A_{j+M/2,r,\alpha}$ represent coefficients and are numerically determined from the solution of a linear system as defined in (Prolo Filho, 2011).

The particular solution to the problem integrated in x, is defined as follows

$$\Psi^p_{xr}(y,\mathbf{\Omega}_i)=\sum_{j=1}^{M/2}\{B_{j,r}(y)\Phi_{xr}(\gamma_{jr},\mathbf{\Omega}_i)+B_{j+M/2,r}(y)\Phi_{xr}(\gamma_{jr},\mathbf{\Omega}_{i+M/2})\}, \tag{37}$$

$$\Psi^p_{xr}(y,\mathbf{\Omega}_{i+M/2})=\sum_{j=1}^{M/2}\{B_{j,r}(y)\Phi_{xr}(\gamma_{jr},\mathbf{\Omega}_{i+M/2})+B_{j+M/2,r}(y)\Phi_{xr}(\gamma_{jr},\mathbf{\Omega}_i)\}, \tag{38}$$

with

$$B_{j,r}(y)=\int_{b_{m-1}}^{y}\left\{\sum_{\alpha=1}^{M}Q_{xr}(\tau,\mathbf{\Omega}_\alpha)B_{j,r,\alpha}\right\}e^{\frac{-(y-\tau)}{\gamma_{jr}}}\,d\tau \tag{39}$$

and

$$B_{j+M/2,r}(y)=-\int_{y}^{b_m}\left\{\sum_{\alpha=1}^{M}Q_{xr}(\tau,\mathbf{\Omega}_\alpha)B_{j+M/2,r,\alpha}\right\}e^{\frac{-(\tau-y)}{\gamma_{jr}}}\,d\tau, \tag{40}$$

where coefficients $B_{j,r,\alpha}$ and $B_{j+M/2,r,\alpha}$ are numerically determined from the solution of a linear system.

Once defined expressions for the homogeneous and particular solutions, the general solution of integrated problems in y and x for a region r, respectively, are given by

$$\Psi_{yr}(x,\mathbf{\Omega}_i)=\Psi^h_{yr}(x,\mathbf{\Omega}_i)+\Psi^p_{yr}(x,\mathbf{\Omega}_i) \tag{41}$$

and

$$\Psi_{xr}(y,\mathbf{\Omega}_i)=\Psi^h_{xr}(y,\mathbf{\Omega}_i)+\Psi^p_{xr}(y,\mathbf{\Omega}_i), \tag{42}$$

for $i=1,\dots,M$ and $r=1,\dots,R$.

To establish the general solution the arbitrary coefficients have to be determined and so it is necessary to solve a linear system, which is constructed using known boundary and interfaces conditions.

The boundary conditions considered here are vacuum on the top and right edges (see Figure (2)),

$$\Psi_r(x,b,\mathbf{\Omega}_i)=0, \qquad i=M/4+1,\dots,M/2, \qquad i=3M/4+1,\dots,M, \tag{43}$$

$$\Psi_r(a,y,\mathbf{\Omega}_i)=0, \qquad i=M/4+1,\dots,M/2, \qquad i=3M/4+1,\dots,M, \tag{44}$$

and reflective on the bottom and left edges (see Figure (2)),

$$\Psi_r(x,0,\mathbf{\Omega}_i)=\Psi_r(x,0,\mathbf{\Omega}_{i+M/4}) \qquad i=1,\dots,M/4, \qquad i=M/2+1,\dots,3M/4, \tag{45}$$

$$\Psi_r(0,y,\mathbf{\Omega}_i)=\Psi_r(0,y,\mathbf{\Omega}_{i+M/4}) \qquad i=1,\ldots,M/4, \qquad i=M/2+1,\ldots,3M/4. \tag{46}$$

In their integrated form

$$\Psi_{xr}(b,\mathbf{\Omega}_i)=0, \qquad i=M/4+1,\ldots,M/2, \qquad i=3M/4+1,\ldots,M, \tag{47}$$

$$\Psi_{yr}(a,\mathbf{\Omega}_i)=0, \qquad i=M/4+1,\ldots,M/2, \qquad i=3M/4+1,\ldots,M, \tag{48}$$

and

$$\Psi_{xr}(0,\mathbf{\Omega}_i)=\Psi_{xr}(0,\mathbf{\Omega}_{i+M/4}) \qquad i=1,\ldots,M/4, \qquad i=M/2+1,\ldots,3M/4, \tag{49}$$

$$\Psi_{yr}(0,\mathbf{\Omega}_i)=\Psi_{yr}(0,\mathbf{\Omega}_{i+M/4}) \qquad i=1,\ldots,M/4, \qquad i=M/2+1,\ldots,3M/4. \tag{50}$$

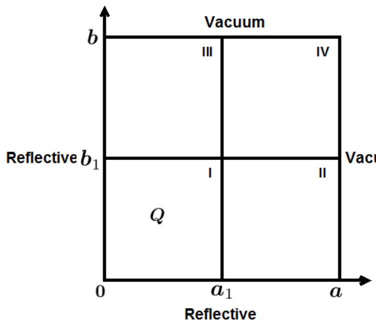

Figura 2: **The domain and its regions.**

3. NUMERICAL RESULTS

We evaluate the region averaged scalar flux, in region r, by the following equations

$$\overline{\phi_r}=\frac{1}{4(a_m-a_{m-1})}\int_{a_{m-1}}^{a_m}\sum_{k=1}^{M/2}w_k[\Psi_{yr}(x,\mathbf{\Omega}_k)+\Psi_{yr}(x,\mathbf{\Omega}_{k+M/2})]dx \tag{51}$$

or

$$\overline{\phi_r}=\frac{1}{4(b_m-b_{m-1})}\int_{b_{m-1}}^{b_m}\sum_{k=1}^{M/2}w_k[\Psi_{xr}(y,\mathbf{\Omega}_k)+\Psi_{xr}(y,\mathbf{\Omega}_{k+M/2})]dy, \tag{52}$$

where $r=1,...,R$. For the cases considered in this work both returned the same values.

We consider in this work the homogeneous problem with isotropic scattering defined in a rectangular domain $a=b=1.0\ cm$ divided into four regions ($r=4$), with fixed isotropic source $Q(x,y)=1.0$ located in the region $[0,0.5]\times[0,0.5]$, as shown Figure (2). The total cross section is $\sigma_t=1.0\ cm^{-1}$ and the scattering cross section is $\sigma_s=0.3\ cm^{-1}$ in for all four regions.

The averaged scalars flux obtained with this formulation are represented in the Tables 1-3. Three approaches were used to approximate the unknown fluxes on the contours, constant, linear and exponential approximations, denoted respectively by: ADO-C, ADO-L and ADO-E. The results are compared with those obtained by AHOT code (Azmy, 2014) based on constant and linear approximations, denoted respectively by: AHOT-C and AHOT-L.

Fixing a number of directions, and analyzing the results obtained for different meshes, we observe up to two significant digits in agreement for the average scalar fluxes, in all regions and for the two quadrature schemes when comparing ADO and AHOT versions.

On the other hand, taking an specific mesh and increasing number of directions by octant, we note a two-digits agreement in the case of the LQ_N scheme. However for the QR quadrature, this convergence achieve up to four digits.

Now, when the comparison is performed between the two methods, considering results obtained with the mesh 4×4, it obtained up to three significant digits, between AHOT-C and ADO-C approaches. For the other approaches, it is observed a reduction of one significant digit. This is observed in all regions and for both quadrature schemes.

Based on this analysis, it is observed that the three approaches proposed in this work, to approximate the unknown fluxes in the contours, ADO-C, ADO-L and ADO-E, have produced satisfactory results in the sense of following the same behavior of the AHOT method. It is also observed that, from a larger division of the domain, the number of significant digits between the average scalar fluxes of both methods begins to increase as would be expected for averaged quantities.

We also verify the influence of using different quadrature schemes, in the sense that with the quadrature QR, we can get more digits of agreement among the numerical results as they increased the quadrature order.

Tabela 1: Averaged scalar flux - Region I, $\sigma_s = 0,3$

	Symmetrical level Quadrature LQ_N											
D[a]	AHOT (Azmy, 2014) Constant			AHOT (Azmy, 2014) Linear			ADO Constant		ADO Linear		ADO Exponential	
	2×2	4×4	50×50	2×2	4×4	50×50	2×2	4×4	2×2	4×4	2×2	4×4
1	0.4562	0.4677	0.4716	0.4768	0.4791	0.4727	0.4627	0.4697	0.5088	0.4806	0.4792	0.4736
3	0.4921	0.4979	0.5004	0.4993	0.5019	0.5007	0.4996	0.5001	0.5494	0.5110	0.5123	0.5031
6	0.4975	0.5029	0.5054	0.5044	0.5067	0.5057	0.5050	0.5051	0.5542	0.5150	0.5171	0.5078
10	0.5004	0.5059	0.5084	0.5072	0.5093	0.5086	0.5078	0.5082	0.5559	0.5172	0.5194	0.5108
21	0.5025	0.5080	0.5104	0.5092	0.5112	0.5106	0.5096	0.5103	0.5560	0.5186	0.5207	0.5128
	Quadrature QR											
D[a]	AHOT (Azmy, 2014) Constant			AHOT (Azmy, 2014) Linear			ADO Constant		ADO Linear		ADO Exponential	
	2×2	4×4	50×50	2×2	4×4	50×50	2×2	4×4	2×2	4×4	2×2	4×4
2	0.4825	0.4857	0.4878	0.4856	0.4872	0.4877	0.4894	0.4878	0.5409	0.4978	0.5043	0.4910
8	0.5055	0.5108	0.5129	0.5114	0.5132	0.5130	0.5120	0.5130	0.5543	0.5210	0.5228	0.5155
18	0.5059	0.5111	0.5133	0.5122	0.5139	0.5135	0.5123	0.5132	0.5546	0.5216	0.5227	0.5157
32	0.5058	0.5110	0.5132	0.5122	0.5139	0.5134	0.5121	0.5131	0.5545	0.5213	0.5224	0.5155

[a] Directions by octant

Tabela 2: Averaged scalar flux - Regions II e III, $\sigma_s = 0,3$

	Symmetrical level Quadrature LQ_N											
D[a]	AHOT (Azmy, 2014) Constant			AHOT (Azmy, 2014) Linear			ADO Constant		ADO Linear		ADO Exponential	
	2×2	4×4	50×50	2×2	4×4	50×50	2×2	4×4	2×2	4×4	2×2	4×4
1	0.1687	0.1607	0.1579	0.1577	0.1553	0.1574	0.1685	0.1603	0.1428	0.1544	0.1640	0.1593
3	0.1717	0.1662	0.1640	0.1612	0.1612	0.1637	0.1716	0.1660	0.1419	0.1610	0.1667	0.1649
6	0.1708	0.1654	0.1637	0.1615	0.1614	0.1634	0.1708	0.1651	0.1415	0.1616	0.1659	0.1642
10	0.1701	0.1648	0.1635	0.1614	0.1613	0.1632	0.1700	0.1645	0.1418	0.1617	0.1654	0.1636
21	0.1694	0.1644	0.1633	0.1613	0.1613	0.1631	0.1694	0.1641	0.1427	0.1616	0.1651	0.1633
	Quadrature QR											
D[a]	AHOT (Azmy, 2014) Constant			AHOT (Azmy, 2014) Linear			ADO Constant		ADO Linear		ADO Exponential	
	2×2	4×4	50×50	2×2	4×4	50×50	2×2	4×4	2×2	4×4	2×2	4×4
2	0.1734	0.1687	0.1671	0.1651	0.1655	0.1669	0.1736	0.1685	0.1433	0.1656	0.1674	0.1673
8	0.1699	0.1657	0.1645	0.1623	0.1624	0.1642	0.1699	0.1655	0.1459	0.1621	0.1658	0.1646
18	0.1688	0.1643	0.1632	0.1612	0.1612	0.1629	0.1687	0.1640	0.1445	0.1609	0.1648	0.1632
32	0.1688	0.1643	0.1631	0.1612	0.1612	0.1628	0.1687	0.1640	0.1445	0.1610	0.1649	0.1632

[a] Directions by octant

Tabela 3: Averaged scalar flux - Region IV, $\sigma_s = 0,3$

	Symmetrical level Quadrature LQ_N											
D[a]	AHOT (Azmy, 2014) Constant			AHOT (Azmy, 2014) Linear			ADO Constant		ADO Linear		ADO Exponential	
	2×2	4×4	50×50	2×2	4×4	50×50	2×2	4×4	2×2	4×4	2×2	4×4
1	0.1299	0.1390	0.1417	0.1176	0.1332	0.1414	0.1312	0.1396	0.1773	0.1493	0.1462	0.1432
3	0.1003	0.1014	0.1010	0.0834	0.0937	0.1005	0.1004	0.1014	0.1358	0.1033	0.1081	0.1026
6	0.0918	0.0932	0.0922	0.0780	0.0869	0.0920	0.0916	0.0932	0.1221	0.0927	0.0979	0.0939
10	0.0882	0.0898	0.0887	0.0750	0.0835	0.0885	0.0879	0.0897	0.1159	0.0889	0.0936	0.0903
21	0.0856	0.0869	0.0863	0.0727	0.0808	0.0859	0.0853	0.0867	0.1116	0.0867	0.0906	0.0874
	Quadrature QR											
D[a]	AHOT (Azmy, 2014) Constant			AHOT (Azmy, 2014) Linear			ADO Constant		ADO Linear		ADO Exponential	
	2×2	4×4	50×50	2×2	4×4	50×50	2×2	4×4	2×2	4×4	2×2	4×4
2	0.0840	0.0827	0.0804	0.0684	0.0754	0.0802	0.0834	0.0825	0.1102	0.0785	0.0898	0.0827
8	0.0830	0.0838	0.0837	0.0691	0.0769	0.0832	0.0828	0.0836	0.1084	0.0858	0.0883	0.0845
18	0.0828	0.0842	0.0833	0.0695	0.0774	0.0830	0.0826	0.0841	0.1086	0.0850	0.0880	0.0849
32	0.0890	0.0843	0.0837	0.0698	0.0777	0.0833	0.0828	0.0841	0.1087	0.0851	0.0881	0.0849

[a] Directions by octant

4. CONCLUDING REMARKS

In this work a two-dimensional fixed source neutron transport problem for homogeneous medium with isotropic scattering, was solved using ADO method in combination with nodal schemes. It has been found that the use of the ADO method is a good alternative for these problems, since allows the use of different numerical schemes of higher-order quadrature maintaining an important characteristic of reducing the order of the associated eigenvalue problem.

The focus of this work was to analyze the effect of using different approaches for appromation unknown flows on the contours. Thus, three different approaches have been proposed: constants, linear and exponential. In the present analysis the influence of the quadrature scheme, in particular the use of higher order schemes, on the scalar fluxes was noted. Better agreement was obtained among the three different ADO approximations in the source region and for more refined meshes. In general such behavior is similar to what happens to the AHOT versions. The complete set of data just indicates one to two digits of possible reference results. The use of refined meshes along with the ADO method is under investigation and it could result in improved analysis. In future works the proposed formulation will be investigated for the solution of radiative transfer applications.

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