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INTEGRAL TRANSFORMATION OF MULTIDIMENSIONAL HEAT CONDUCTION PROBLEMS IN HETEROGENEOUS MEDIA

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Abstract. *The integral transform method is applied in the analytical solution of three-dimensional transient heat conduction problems in heterogeneous media, with the heterogeneities being represented by arbitrarily space variable coefficients. The Generalized Integral Transform Technique (GITT) is employed in the solution of multidimensional Sturm-Liouville eigenvalue problem, by adopting a simpler auxiliary eigenvalue problem and proposing an expansion for the original eigenfunctions. Also, in combination with a single domain formulation strategy, the proposed solution also handles multiregion and multiscale problem formulations. The methodology is demonstrated for two test cases. The first example is related to Functionally Graded Materials (FGM), with multiscale variations of thermal conductivity and capacitance in a three-dimensional domain. The second example involves abrupt variations of thermophysical properties simulating the interface among multiple materials through heat conduction in a composite matrix with spherical and cylindrical dispersed elements. The computational simulations here presented have been implemented in the symbolic-numeric Mathematica platform, version 11.1. Simulations in COMSOL Multiphysics, version 4.4, are also employed for critical comparisons against the GITT solutions.*

Keywords: *Heat conduction, Integral transforms, GITT, Heterogeneous media, FGM, Composite medium, Eigenvalue problem, Single domain formulation.*

1. INTRODUCTION

Problems of heat and mass transfer in heterogeneous media are commonly encountered in different contexts of physics and engineering. According to Lin (1992), transient conduction in chemical catalysts, heat exchangers and electronic systems are practical examples of such problems. One of the main difficulties in modeling this type of problem is that thermophysical properties vary across the domain, either abruptly or gradually. In this case, we highlight the complexity of finding a simple physical model that represents the transient process of heat transfer in such complex domains. In recent years, interest in the analysis of heat conduction in heterogeneous media has been renewed under the motivation of developments in the manufacturing of new materials, such as functionally graded materials (FGM) and composites.

According to Callister (1999), aerospace, underwater and transport applications have been requiring materials with unusual properties, which cannot be achieved by metallic alloys, ceramics and conventional polymeric materials. The combination of materials towards challenging ranges of properties were, and still are, being expanded through the study of composite materials. In general, composite is considered a multiphase material that exhibits a significant proportion of the constituent phases, obtaining a desired combination of properties.

A general integral transforms solution for heat conduction in heterogeneous media has been proposed by Naveira-Cotta *et al.* (2009), with the heterogeneities being represented by arbitrarily variable space coefficients, including thermophysical properties and source terms. This analytical approach was then employed in the direct and inverse analysis of heat conduction problems in heterogeneous media (Naveira-Cotta *et al.* 2009; Naveira-Cotta *et al.* 2011; Knupp *et al.* 2012; Knupp et al., 2013), including situations of FGM, nanocomposites with abrupt variations of thermophysical properties, and randomly distributed two-phase dispersed systems. The computational procedure involves the solution of the associated Sturm-Liouville problem with space variable coefficients through the Generalized Integral Transform Technique (GITT) (Cotta, 1993; Cotta and Mikhailov, 1997; Cotta et al., 2016a), a hybrid numerical-analytical approach that transforms the differential eigenvalue problem into an algebraic matrix eigenvalue problem. Such developments induced the proposition of a single domain reformulation strategy, originally developed in the context of conjugated heat

transfer problems (Knupp et al., 2012; Knupp et al., 2015a; Knupp et al., 2015b), which consists of rewriting a multiregion problem into a single region with space variable thermophysical properties and source terms, and is particularly convenient for handling problems with multiple regions and different materials. Recently, a convergence acceleration scheme has been developed and applied to one-dimensional eigenvalue problems (Cotta et al., 2016b), which allowed for the treatment of multiscale space variations, due to either abrupt variation of thermophysical properties or to multiple regions with markedly different geometrical sizes.

The present work further demonstrates the integral transforms solution by implementing a symbolic-numerical computational algorithm for the analysis of transient three-dimensional heat conduction in heterogeneous media. The general Sturm-Liouville problem is solved by considering the simplest possible auxiliary eigenvalue problem with constant coefficients. Numerical results are then reported for a couple of three-dimensional tests cases, one associated with a FGM application (Naveira-Cotta *et al.* 2009) and the other involving abrupt variations in multiple materials within a matrix filler. A thorough convergence analysis is here provided, so as to illustrate the integral transforms solution computational performance.

2. PROBLEM FORMULATION AND SOLUTION METHODOLOGY

The computational algorithm was built on the *Mathematica* platform, v.11.1, based on the following transient three-dimensional diffusion formulation:

$$w(x,y,z)\frac{\partial T(x,y,z,t)}{\partial t} = \nabla.[k(x,y,z)\nabla T(x,y,z,t)] - d(x,y,z)T(x,y,z,t) + g(x,y,z,t,T),$$

$$x_0 < x < x_1,\ y_0 < y < y_1,\ z_0 < z < z_1,\ t > 0 \tag{1.a}$$

$$\alpha_{x,l}T(x_l,y,z,t)+(-1)^{l+1}\beta_{x,l}k(x_l,y,z)\frac{\partial T(x,y,z,t)}{\partial x}\bigg|_{x=x_l}=0, \qquad l=0,1,$$

$$y_0 < y < y_1, \qquad z_0 < z < z_1, \qquad t > 0 \tag{1.b, c}$$

$$\alpha_{y,l}T(x,y_l,z,t)+(-1)^{l+1}\beta_{y,l}k(x,y_l,z)\frac{\partial T(x,y,z,t)}{\partial y}\bigg|_{y=y_l}=0, \qquad l=0,1,$$

$$x_0 < x < x_1, \qquad z_0 < z < z_1, \qquad t > 0 \tag{1.d, e}$$

$$\alpha_{z,l}T(x,y,z_l,t)+(-1)^{l+1}\beta_{z,l}k(x,y,z_l)\frac{\partial T(x,y,z,t)}{\partial z}\bigg|_{z=z_l}=0, \qquad l=0,1,$$

$$x_0 < x < x_1, \qquad y_0 < y < y_1, \qquad t > 0 \tag{1.f, g}$$

$$T(x,y,z,t)=f(x,y,z), \qquad t=0, \qquad x_0 < x < x_1, \qquad y_0 < y < y_1, \qquad z_0 < z < z_1 \tag{1.h}$$

The above formulation assumes that a filtering solution, either explicit or implicit, has already been applied so as to make the boundary conditions homogeneous in all the spatial coordinates. Also, it is sufficiently general to incorporate nonlinearities in the equation source terms, which can be originated from the source itself or from nonlinear operators not represented in the rest of the diffusion equation. Following the formalism in the integral transform method (Naveira-Cotta *et al.*, 2009; Cotta, 1993; Cotta and Mikhailov, 1997), the integral transform pair is proposed as:

$$\overline{T}_i(t) = \int_{z=z_0}^{z_1}\int_{y=y_0}^{y_1}\int_{x=x_0}^{x_1} w(x,y,z)\,\tilde{\psi}_i(x,y,z)\,T(x,y,z,t)\,dxdydz, \quad \text{Transform} \tag{2.a}$$

$$T(x,y,z,t) = \sum_{i=1}^{\infty}\tilde{\psi}_i(x,y,z)\,\overline{T}_i(t), \quad \text{Inverse} \tag{2.b}$$

where the normalized eigenfunction and the normalization integral are given by

$$\tilde{\psi}_i(x,y,z) = \frac{\psi_i(x,y,z)}{\sqrt{N_i}}; \quad N_i = \int_{z=z_0}^{z_1}\int_{y=y_0}^{y_1}\int_{x=x_0}^{x_1} w(x,y,z)\psi_i^2(x,y,z)dxdydz \tag{3.a, b}$$

The eigenfunctions $\psi_i(x, y, z)$ and the corresponding eigenvalues μ_i can be evaluated from the following three-dimensional Sturm-Liouville problem:

$$\nabla.[k(x, y, z)\nabla\psi_i(x, y, z)]+[\mu_i^2 w(x, y, z) - d(x, y, z)]\psi_i(x, y, z) = 0 \tag{4.a}$$

$$\alpha_{x,l}\psi_i(x_l, y, z) + (-1)^{l+1}\beta_{x,l}k(x_l, y, z)\frac{\partial\psi_i(x, y, z)}{\partial x}\bigg|_{x=x_l} = 0, \qquad l = 0,1,$$

$$y_0 < y < y_1, \qquad z_0 < z < z_1, \qquad t > 0 \tag{4.b, c}$$

$$\alpha_{y,l}\psi_i(x, y_l, z) + (-1)^{l+1}\beta_{y,l}k(x, y_l, z)\frac{\partial\psi_i(x, y, z)}{\partial y}\bigg|_{y=y_l} = 0, \qquad l = 0,1,$$

$$x_0 < x < x_1, \qquad z_0 < z < z_1, \qquad t > 0 \tag{4.d, e}$$

$$\alpha_{z,l}\psi_i(x, y, z_l) + (-1)^{l+1}\beta_{z,l}k(x, y, z_l)\frac{\partial\psi_i(x, y, z)}{\partial z}\bigg|_{z=z_l} = 0, \qquad l = 0,1,$$

$$x_0 < x < x_1, \qquad y_0 < y < y_1, \qquad t > 0 \tag{4.f, g}$$

The GITT (Naveira-Cotta *et al.* 2009; Naveira-Cotta *et al.*, 2011; Knupp *et al.*, 2012; Cotta, 1993) can be employed to solve the eigenvalue problem defined by Eqs. (4.a-g), based on a simpler auxiliary eigenvalue problem. The simplest possible choice of auxiliary problem with constant coefficients has been here adopted, as will be seen in the examples to follow. The integral transformation of Eqs. (1.a-h) leads to the following transformed system:

$$\frac{d\bar{T}_i(t)}{dt} + \mu_i^2\bar{T}_i(t) = \bar{g}_i(t, \bar{T}_j(t)), \quad t > 0, \ i, j = 1, 2, 3, ... \tag{5.a}$$

$$\bar{T}_i(0) = \bar{f}_i = \int_{z=z_0}^{z_1} \int_{y=y_0}^{y_1} \int_{x=x_0}^{x_1} w(x, y, z)f(x, y, z)\tilde{\psi}_i(x, y, z)dxdydz \tag{5.b}$$

where the transformed source term is given by

$$\bar{g}_i(t, \bar{T}_j(t)) = \int_{z=z_0}^{z_1} \int_{y=y_0}^{y_1} \int_{x=x_0}^{x_1} g(x, y, z, t, T)\tilde{\psi}_i(x, y, z)dxdydz \tag{5.c}$$

System (5) can be numerically solved for the transformed potentials employing the routine NDSolve from the *Mathematica* platform, with automatic control of absolute and relative errors. For a linear problem, the temperature field can be readily obtained analytically solving system (5) and employing the inverse formulae Eq. (2.b), to yield:

$$T(x, y, z, t) = \sum_{i=1}^{\infty}\tilde{\psi}_i(x, y, z)[\bar{f}_i e^{-\mu_i^2 t} + \int_0^t \bar{g}_i(t')e^{-\mu_i^2(t-t')}dt'] \tag{6}$$

In both tests-cases solved here, one associated with a FGM application (Naveira-Cotta *et al.*, 2009) and the other related to multiple non-contacting three-dimensional regions within a matrix filler, the boundary coefficients present in Eqs. (1,4) were kept fixed as:

$$\alpha_{x,0} = \alpha_{y,0} = \alpha_{z,0} = 0, \ \beta_{x,0} = \beta_{y,0} = \beta_{z,0} = 1, \ \alpha_{x,1} = \alpha_{y,1} = \alpha_{z,1} = 1, \ \beta_{x,1} = \beta_{y,1} = \beta_{z,1} = 0$$

2.1 Test-Case 1: FGM with properties variation in x, y, and z directions

The first test case deals with heat conduction in a FGM, governed by Eqs. (1), in dimensionless form, with thermophysical properties varying in the x, y, z directions. The coefficients of Eqs. (1.a, h) are given by:

$$k(x, y, z) = k_0 e^{[2b(x+y+z)]}, \qquad w(x, y, z) = w_0 e^{[2b(x+y+z)]}, \qquad d(x, y, z) = 0,$$

$$f(x, y, z) = 1, \qquad g(x, y, z, t, T) = 0$$

(7.a, e)

where the required numerical coefficients are chosen as $k_0 = 1$, $w_0 = 10$, $b = 1$ (Naveira-Cotta *et al.*, 2009).

2.2 Test-Case 2: Composite with spherical and cylindrical dispersed elements

The second test case deals with heat conduction in a heterogeneous medium with spherical and cylindrical dispersed elements within matrix filler material, all of known physical properties. The physical model of the second test case is shown in Figure 1. As can be observed, the central cylinder is rotated of 45° with respect to the vertical axis z.
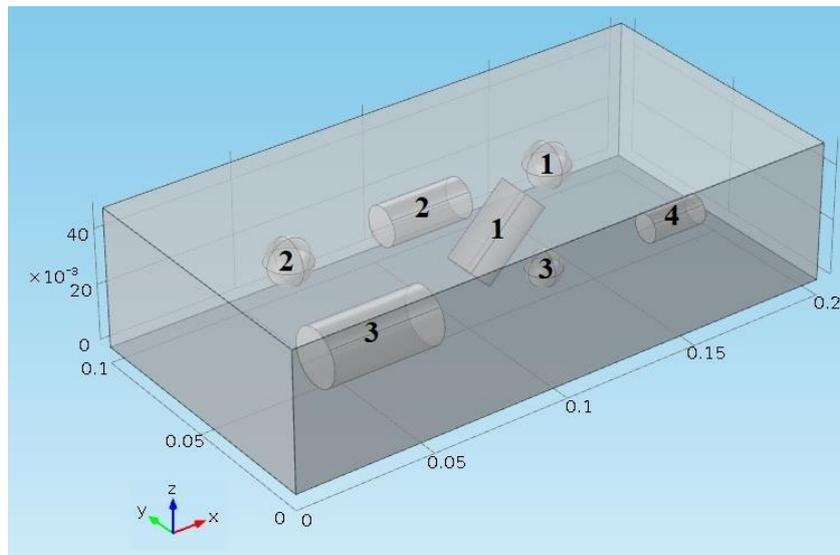

Figure 1. Composite medium consisting of dispersed elements formed by three spheres and four cylinders surrounded by a continuum parallelepiped matrix.

The thermophysical properties for the single domain formulation are given by:

$$w(x, y, z), \ k(x, y, z) = \begin{cases} w_1, \ k_1, & \text{Cylinders and Spheres (dispersed phase)} \\ w_2, \ k_2, & \text{Parallelepiped (continuum phase)} \end{cases}$$

(8.a, b)

The dimensions of the parallelepiped matrix were defined as: $L_x = 0.2$ m, $L_y = 0.1$ m and $L_z = 0.05$ m. Furthermore, the dimensions of the dispersed elements are presented in Tables 1 and 2.

Table 1. Dimensions of the Cylinders.

Cylinder	Height (mm)	Radius (mm)	z-angle (°)	Position (x,y,z) mm
1	30	8	45	(100,50,15)
2	30	7	0	(85,75,25)
3	40	10	0	(25,25,25)
4	20	5	0	(150,25,25)

Table 2. Dimensions of the Spheres.

Sphere	Radius (mm)	Position (x,y,z) mm
1	8	(150,75,25)
2	8	(50,75,25)
3	6	(110,25,25)

Table 3 shows the thermophysical properties of the materials used in this test case. The parallelepiped matrix phase is made of a copper alloy with aluminum (component 2), while the cylindrical and spherical dispersed phases (component 1) are made of dura-aluminum (aluminum - copper alloy) (Kreith *et al.*, 2011).

Table 3. Thermophysical properties and initial conditions of the composite consisting of metallic dispersed elements
formed by three spheres and four cylinders surrounded by a metal matrix filler (Kreith *et al.*, 2011).

Component	Phase	Materials	Composition	$k\left(\dfrac{W}{m\,K}\right)$	$w \times 10^6 \left(\dfrac{J}{m^3 K}\right)$	Initial Temp. (°C)
1	Dispersed	Duraluminum	94-96% Al, 3-5% Cu trace Mg	164	2.32	50
2	Continuum	Copper alloy w/aluminum	95% Cu, 5% Al	83	3.55	50

3. RESULTS

The input data used in the present study for both the FGM and the composite medium with three-dimensional inserts, were presented in the previous section. Although the formulated test cases are linear ones, the NDSolve routine, available in the *Mathematica* platform, was used in its default mode, which considers automatic control to eight digits for the AccuracyGoal and PrecisionGoal controllers. In practice, the AccuracyGoal parameter specifies the local absolute error allowed in each step of the numerical solution, whereas the PrecisionGoal parameter specifies the relative error.

The COMSOL implementation was performed following the steps: (a) definition of constants, presented in Table 3; (b) geometry of the composite medium; (c) choosing the physics of the problem. For the present test cases, it was chosen heat transfer in solids and transient regime; (d) meshing; (e) results interpretation.

3.1 Results for FGM with properties variation in x, y, z directions

The parameters used in this simulation were shown in Eqs. (7.a,e) assuming the following values for the coefficients: $k_0 = 1$, $w_0 = 10$, $b = 1$ (Naveira-Cotta *et al.*, 2009). This problem was solved in dimensionless form, with the dimensions of the domain chosen as: $L_x = L_y = L_z = 1$. Figures 2 and 3 illustrate the variation of the properties along the (x,y) plane for z = 1. We note that a ratio of approximately 400 times is achieved between the values of the coefficients k(x,y,z) and w(x,y,z) in opposite boundaries.
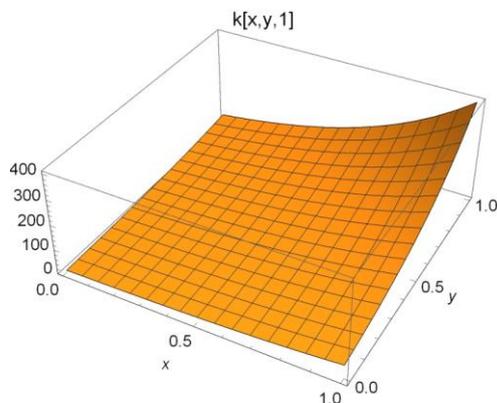

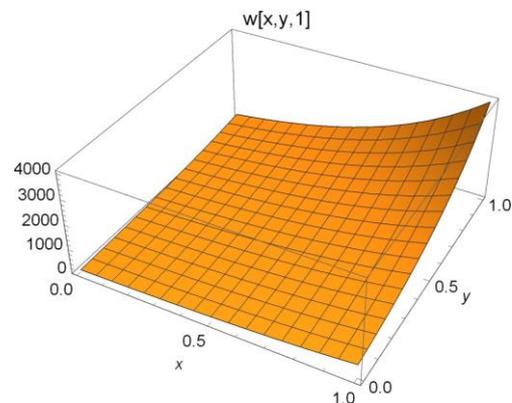Figure 2. Variation of the dimensionless thermal conductivity coefficient $k(x,y,z)$ for the FGM at z = 1.

Figure 3. Variation of the dimensionless thermal capacity coefficient $w(x,y,z)$ for the FGM at z = 1.

Table 4 shows a brief convergence analysis of the eigenvalue problem solved by the GITT in this first test case. We can observe the convergence to five significant digits for truncation orders of the eigenvalue problem around 200 terms. Figure 4 shows the estimated relative errors of the eigenvalues for increasing truncation orders. For truncation orders (N_T) less than 200 terms the estimated relative deviations on the selected eigenvalues were less than 0.01%, indicating the excellent convergence attained in the solution of the eigenvalue problem with space variable coefficients. These relative errors were calculated with respect to the highest truncation order, $N_T = 230$, according to:

$$\in_{\mu} = \left| \frac{\mu_n - \mu_{n_{max}}}{\mu_n} \right| .100 \tag{9}$$

where $\mu_{n_{max}}$ represents the eigenvalue of order n with greatest truncation order and $\in_{\mu}$ represents the percent relative error. Table 5 presents the temperature convergence analysis for different values of the coordinate x, at y = 0.4, z = 0.4,

and t = 0.2, where M_T is the truncation order for the temperature expansion. One can notice five converged significant digits in all axial positions. Also, it can be verified the adherence to at least two significant digits through GITT and the NDSolve routine of *Mathematica* 11.1 (Num. Sol.), in its default mode for directly solving the partial differential equation. The largest relative deviation in the evaluated points was 0.19%, when using 200 terms in the expansion for the eigenvalues. This deviation was calculated through Eq. (10) below and the comparison between the solutions is shown in Fig. 5, where an excellent agreement is demonstrated, for two different times (t=0.3 and t=0.6), along the x-coordinate.

$$Deviation\ (\%) = \left| \frac{T_{GITT}(x,y,z,t) - T_{NDSolve}(x,y,z,t)}{T_{NDSolve}(x,y,z,t)} \right| \cdot 100 \tag{10}$$

Table 4. Convergence analysis of the eigenvalues obtained through GITT for the FGM example.

| μ_i	$N_T = 50$	$N_T = 80$	$N_T = 110$	$N_T = 140$	$N_T = 170$	$N_T = 200$	$N_T = 230$
1	1.2390	1.2390	1.2389	1.2389	1.2389	1.2389	1.2389
2	1.8810	1.8810	1.8809	1.8808	1.8808	1.8808	1.8808
3	1.8816	1.8810	1.8809	1.8809	1.8808	1.8808	1.8808
4	1.8816	1.8810	1.8809	1.8809	1.8808	1.8808	1.8808
5	2.3549	2.3539	2.3539	2.3539	2.3537	2.3537	2.3537
6	2.3549	2.3539	2.3539	2.3539	2.3537	2.3537	2.3537
7	2.3549	2.3539	2.3539	2.3539	2.3537	2.3537	2.3537
8	2.7378	2.7378	2.7369	2.7369	2.7369	2.7369	2.7367
9	2.7378	2.7378	2.7369	2.7369	2.7369	2.7369	2.7369
10	2.7378	2.7378	2.7369	2.7369	2.7369	2.7369	2.7369
20	3.3933	3.3920	3.3913	3.3906	3.3906	3.3905	3.3905
30	3.9601	3.9277	3.9275	3.9274	3.9265	3.9265	3.9265
40	4.4319	4.4040	4.4025	4.4020	4.4020	4.4012	4.4011
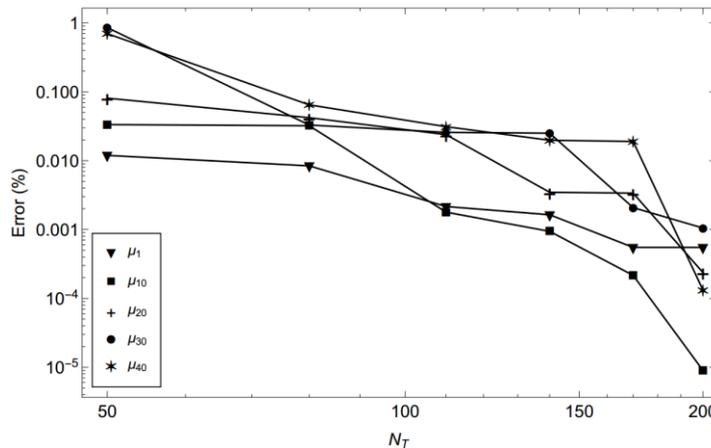
Figure 4. Relative errors of 1st, 10th, 20th, 30th, and 40th eigenvalues for the FGM example eigenvalue problem.

Table 5. Temperature convergence along the dimensionless axial coordinate x at y = 0.4, z = 0.4 and t = 0.2, using 200 terms in the eigenvalue problem.

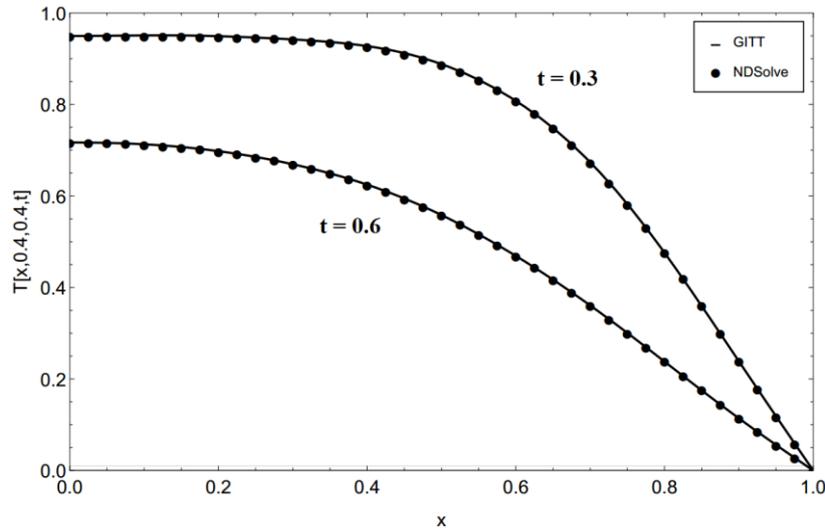| T(x,0.4,0.4,0.2)					
M_T	0	0.2	0.4	0.6	0.8
10	0.8705	0.8733	0.9645	1.0031	0.6456
40	0.9771	0.9899	0.9805	0.9226	0.6057
70	0.9930	0.9922	0.9916	0.9256	0.6116
100	0.9902	0.9926	0.9873	0.9246	0.6110
130	0.9902	0.9926	0.9879	0.9246	0.6111
160	0.9903	0.9925	0.9879	0.9246	0.6110
190	0.9903	0.9925	0.9879	0.9246	0.6110
Num. Sol. (NDSolve)	0.9903	0.9903	0.9856	0.9242	0.6108
Deviation (%)	0	0.19	0.18	0.035	0.0084

Figure 5. Comparison between the GITT solution (using 200 terms in the eigenvalue problem and 70 terms in the temperature expansion) and the fully numerical solution (NDSolve) of the partial differential equation, along the x coordinate, at y = 0.4, z = 0.4, for times t = 0.3 and t = 0.6.

3.2 Results for Composite with Spherical and Cylindrical Dispersed Elements

The physical model employed in this test case was presented in Figure 1. To perform the integration of the coefficients in the GITT approach, either numerical or analytic, in a specific region, a new function in the *Mathematica* platform has been employed. The function performs a change of variables and remaps the defined region over a unit cube, which can be integrated with the function option *MultiDimensionalRule* from the *Mathematica* system. In order to use the numerical integration over regions, it is first necessary to create a region through the command *Region*.

Here, two distinct regions were created, the first one formed by cylinders and the other formed by spheres. After that, one can use the following command line to evaluate the integral over a specific region:

$$I = \text{NIntegrate}\left[f\left(x, y, z\right), \{x, y, z\} \in \text{Region}, \text{Method} \rightarrow \text{"MultidimensionalRule"} \right] \tag{11}$$

where f(x,y,z) can be, for instance, the eigenfunction, integrated over the *Region* previously defined. In the solution of this test case, the *NIntegrate* function was employed with the integration method called *MultiDimensionalRule*, for the evaluation of the integrals in the cylindrical and spherical regions.

To solve the integrals of the coefficients that compose the algebraic matrix eigenvalue problem (Cotta, 1993; Cotta and Mikhailov, 1997) the regions were here treated separately, being analytically integrated in the parallelepiped, including the sphere and cylindrical regions, and numerically, through the function NIntegrate, in the spherical and cylindrical regions. Then, the subtraction of these results is performed, recalling the properties of each material. Also, the parameters PrecisionGoal and AccuracyGoal were set equal to 6 and 4, respectively, to perform the integrals that contain the thermal capacity term w(**x**) and the thermal conductivity term k(**x**).

In this test case, the solution obtained through the GITT was critically compared with the numerical solution obtained via the COMSOL Multiphysics 4.4 software. In the COMSOL numerical simulations, the numerical values of 10^{-3} and 10^{-4} were set for the relative and absolute tolerances, respectively. Moreover, the mesh in COMSOL was built with the automatic mesh control, informing only the maximum element size of the generated mesh. As we can see in Figure 6, the elements of the mesh with maximum size of 30 mm are sufficient to ensure the convergence of the solution to the graphical scale, as compared to the fully converged GITT solution.

Table 6 summarizes the tested meshes maximum sizes and the number of elements utilized in each of them. Also, Figure 7 illustrates the final configuration of the finite element mesh with a maximum element size of 30 mm.

Table 7 shows the convergence analysis of the eigenvalue problem solved by GITT for this second test case. One can notice the convergence to four significant digits for the selected eigenvalues at a truncation order around $N_T = 320$.

Figure 8 shows the decrease on the relative errors of the eigenvalues for increasing truncation orders in the eigenfunction expansions. For truncations orders (N_T) less than 260 terms the relative deviation was already less than 0.05%, indicating an excellent convergence rate of the eigenvalue problem. These relative errors were calculated with respect to the largest truncation order for the expansions, that is, for $N_T = 320$, according Eq. (9).
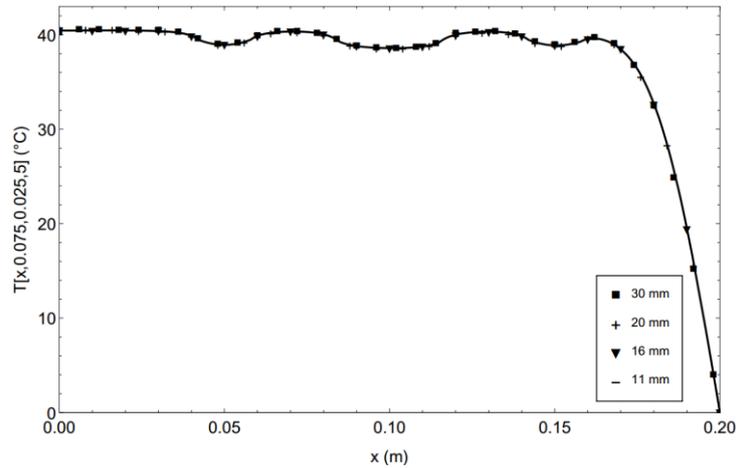
Figure 6. Comparison of temperature profiles against converged GITT solution (solid line), at the point y = 0.075 m, z = 0.025 m, and t = 5 s, as obtained from the COMSOL numerical solution with different maximum mesh sizes (symbols) for the composite medium example.

Table 6. Maximum mesh element size and number of elements in domain for composite medium example.

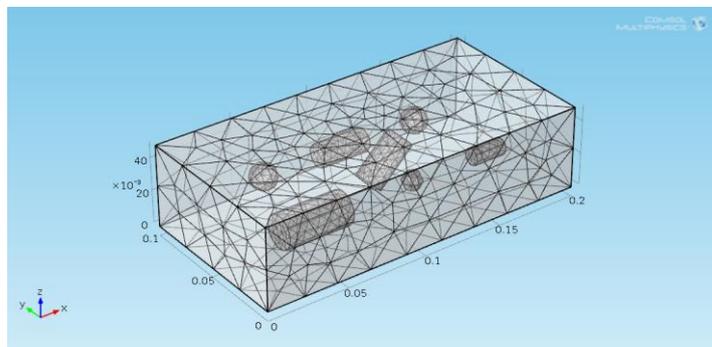| Maximum element size (mm)	Total number of elements in the domain
30	11533
20	20584
16	32794
11	51956

Figure 7. Final configuration of the mesh with maximum element size of 30 mm for composite medium example.

Table 7. Convergence analysis of the eigenvalues obtained through GITT for the composite medium example

| μ_i	$N_T = 80$	$N_T = 140$	$N_T = 200$	$N_T = 260$	$N_T = 320$
1	0.1784	0.1783	0.1782	0.1782	0.1782
2	0.2083	0.2082	0.2082	0.2081	0.2081
3	0.2595	0.2594	0.2593	0.2593	0.2593
4	0.2824	0.2821	0.2820	0.2819	0.2819
5	0.3013	0.3012	0.3011	0.3010	0.3010
6	0.3218	0.3216	0.3215	0.3214	0.3214
7	0.3379	0.3377	0.3376	0.3376	0.3375
8	0.3875	0.3873	0.3871	0.3871	0.3870
9	0.3905	0.3900	0.3898	0.3896	0.3896
10	0.4198	0.4193	0.4190	0.4188	0.4187
20	0.5194	0.5190	0.5187	0.5186	0.5185
30	0.5874	0.5865	0.5859	0.5855	0.5853
40	0.6425	0.6415	0.6411	0.6409	0.6407
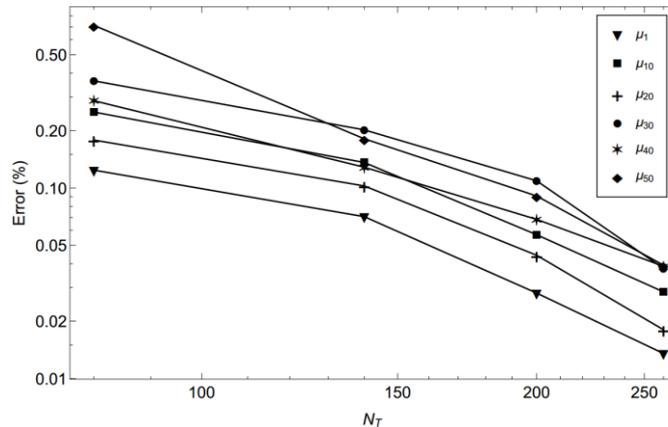
Figure 8. Estimated relative errors of the 1st, 10th, 20th, 30th, 40th and 50th eigenvalues of the composite medium eigenvalue problem.

Table 8 presents the convergence analysis of the temperature field for different values of x, at y = 0.075 m, z = 0.025 m, and t = 5 s, where M_T is the truncation order of the temperature eigenfunction expansions. One can notice four fully converged significant digits in all axial positions presented. The largest relative deviation in all the evaluated points was 0.48%, when using 260 terms in the expansion for the eigenvalues. This relative error was calculated through Eq. (10). The comparison between the two solutions, GITT and COMSOL, can also be visualized to the graphical scale in Figure 9, where a good agreement is demonstrated along the x-coordinate x.

Table 8. Temperature convergence along the axial coordinate x at y = 0.075 m, z = 0.025 m and t = 5 s, using 260 terms in the eigenfunction expansion, for the composite medium example.

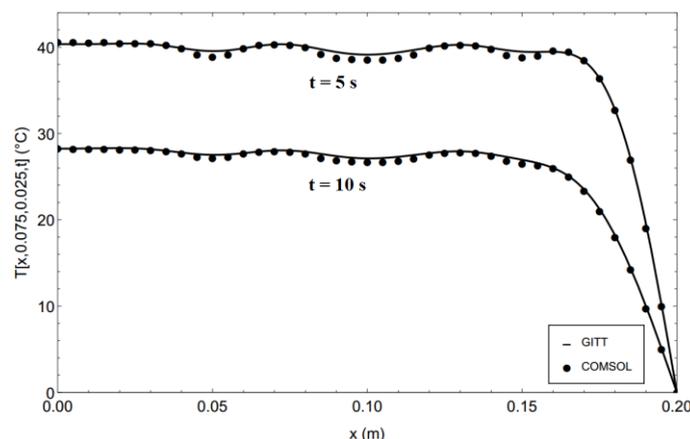|	T(x,0.075,0.025,5)				
M_T	0	0.04	0.08	0.12	0.16
10	37.19	36.44	35.61	33.60	33.94
40	40.92	40.30	40.82	40.79	40.67
70	40.56	40.09	40.39	40.31	39.92
100	40.36	39.90	40.10	39.97	39.47
130	40.36	39.88	40.06	39.94	39.49
160	40.32	39.84	40.02	39.92	39.49
190	40.29	39.82	40.02	39.91	39.50
220	40.30	39.82	40.01	39.91	39.48
250	40.30	39.82	40.01	39.90	39.48
Num. Sol. (COMSOL)	40.25	39.95	40.08	40.03	39.67
Deviation (%)	0.12	0.32	0.17	0.32	0.48

Figure 9. Comparison between the GITT and COMSOL solutions in the composite medium example, along the x coordinate, at y = 0.075 m, z = 0.025 m, and for times t = 5 s and t = 10 s. For the GITT solution it was employed 260 terms in the eigenfunction expansion and 220 terms in the temperature expansion.

4. CONCLUSIONS

Three-dimensional heat conduction in heterogeneous media was analyzed through the integral transform method, for fairly different situations characterized by both continuous but multi-scale or abrupt discontinuous variations of the diffusion equation space variable coefficients. First, the transient heat conduction problem in a FGM with thermophysical properties varying simultaneously in x, y, and z directions was analyzed, followed by an example dealing with a composite media formed by a matrix filler material and with dispersed elements composed by spheres and cylinders. The single domain formulation strategy was applied to rewrite the physical properties as space variable coefficients within one single diffusion equation for the whole domain. In the first test case, the solution obtained through GITT is critically compared with the numerical solution obtained through the NDSolve function of the *Mathematica* platform. The second test case was solved both through GITT and the COMSOL Multiphysics 4.4. Recent updates of the symbolic computation platform *Mathematica* have added interesting features that can be employed in solving a wide range of convection-diffusion problems through integral transforms. Among these new features, the integration over regions can be quite useful in dealing with heterogeneous media with different geometries of sub-regions, as here demonstrated. For both tests cases here implemented, an excellent agreement between the hybrid and numerical solution methodologies was achieved.

5. ACKNOWLEDGEMENTS

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