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Asymptotic and numerical solutions of the Orr-Sommerfeld equation for surface waves on an inclined plane.

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Abstract: *This work presents an analysis of the initial behavior of free surface liquid flows on inclined planes where, under some conditions, long-wave instability may appear. These instabilities may evolve to surface-wave, that often appear on thin liquid films. Such knowledge is useful in industry, once liquid films help to remove the heat from solid surfaces, and also reduces the friction between high viscosity fluids and pipe walls, by injecting, close to the wall, a less viscous fluid. The surface-wave instability is governed by the Orr-Sommerfeld equation and their boundary conditions. In this work, we present a long-wave solution for the Orr-Sommerfeld equation based on asymptotic expansions and a numerical solution. For the long-wave perturbations, the wave number can be treated as a small parameter, and the form of the equations suggests that the speed and amplitude of the eigenfunction can be sought as a power series of the wave number, which we considered until the second order of approximation for the asymptotic solution. The numerical solution was based on a Galerkin method using Chebyshev polynomials for the discratization, which made it possible to express the Orr-Sommerfeld equation and their boundary conditions as a generalized eigenvalue problem. Those choices were made because of the general approach of the Galerkin method, which makes the implementation of the boundary condition of free surface easier, and the high accuracy of the Chebyshev polynomials. A code was implemented in Matlab to solve the linear system using a QZ algorithm. The asymptotic and numerical solutions give an approximation for the physical eigenvalue of the problem. Once in possession of this result it is possible to find the growth rate, the wave speed, the wavelength and the critical Froude number of the liquid film at the instability threshold. The results are compared with previously published data.*
Keywords: *Surface waves, Orr-Sommerfeld equation, Asymptotic method, Chebyshev polynomials, Galerkin method.*

1. Introduction.

This work is devoted to present a study about surface waves instabilities in liquid films on an inclined plane. Industrial applications such as friction reducing effects and reactors cooling process are some examples, therefore a good knowledge about this problem is central to industrial applications. When dealing with such flows, the appearance of instabilities, in this case surface waves, must be considered. Depending on the system being used, the presence of these instabilities can be (or not) desirable. Considering the system of oil transport in an annular flow, the film thickness must be constant, therefore it is necessary to avoid instability. On the other hand, sometimes we can use instabilities to increase heat and mass transfer, for example, in a reactor cooling process. This article present a study on some physical aspects of a liquid film falling down on a inclined plane without heat exchange with the wall, and no surfactants at interface. In the first section we considered some physical properties of the problem and their pertinent dimensionless groups, then we use the perturbations in the form of a stream function as done by (Yih, 1963). Considering a linearization of the perturbations in the Navier-Stokes equation, we have the equation of Orr-Sommerfeld (Orr, 1907; Sommerfeld, 1908). The second section is devoted to solve the generalized eigenvalue problem, produced by the Orr-Sommerfeld equation together with the flow boundary conditions, by means of an asymptotic analysis (Kevorkian and Cole, 1981), similar to the approach employed by (Smith, 1990), who considered the wave number as a small parameter and dealt with the speed and amplitude of the waves as an expansion in power series of the characteristic wave number. For the numerical solution we implemented a Galerkin method (Fletcher, 1984) using Chebyshev polynomials for the discretization of the equations (Boyd, 1989). In this work we performed the analytical expansions up to the second order, we found the growth rate of the instability, the wave speed, and critical Froude number. At the end we compared the results with the numerical solution for validation.

2. Formulation of the problem.

We consider a film of incompressible Newtonian liquid of thickness h, viscosity μ and density ρ, falling down on a inclined plane with an angle θ with respect to the horizontal and without presence of a gradient temperature (Fig. 1) . The interface between the liquid and the gas has a surface tension γ and no surfactants present. The pressure applied by the gas on the interface is P_0. Based on the Navier-Stokes equations and considering the boundary conditions of permanent flow, non-slip at solid surface, no shear and constant pressure at the liquid-gas interface, we can find a solution corresponding to a steady parallel flow with a planar interface and parabolic velocity profile:

$$\overline{U}(y) = U_0(1 - \frac{y^2}{h^2}) \tag{1}$$

$$\overline{V}(y) = 0 \tag{2}$$

$$\overline{P}(y) = P_0 - \rho g cos(\theta) y \tag{3}$$

where the velocity of the interface U_0 is given by,

$$U_0 = \frac{\rho g h^2 sin(\theta)}{2\mu} \tag{4}$$

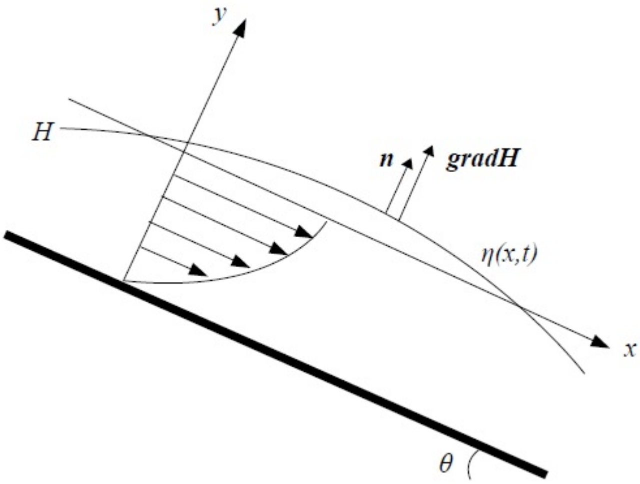

Figura 1: Liquid film falling down on a inclined plane with parabolic velocity profile, planar interface H, where $\vec{n}$ is parallel to the gradient of H (∇H), and perturbed interfacial position $\eta(x,t)$ (Chimetta and Franklin, 2015).

The solutions are given in terms of the parameters h, ρ and U_0, where the latter is the speed of the fluid at the interface. The contribution of inertia relative to viscosity, gravity, and surface tension is measured, respectively, by the Reynolds, Froude, and Weber numbers, defined as:

$$Re = \frac{\rho U_0 h}{\mu}, \ \ Fr = \frac{U_0^2}{g h cos(\theta)} = \frac{Re \ \tan(\theta)}{2}, \ \ We = \frac{\rho U_0^2 h}{\gamma} \tag{5}$$

where the Froude number is defined using the gravity component $gcos(\theta)$ normal to the flow. When the interface is perturbed ($\eta(x,t) \neq 0$), the velocity profile no longer has a exactly parabolic behaviour. This, combined with inertia, leads to surface waves instabilities.

2.1 The equation of Orr-Sommerfeld.

The Squire's theorem (Squire, 1933), sets that the first instability on parallel flow of a Newtonian fluid, is always two-dimensional. This result remains valid for a flow with interface (Hesla *et al.*, 1986), then we can consider only the two-dimensional case of the problem. We will use the Navier-Stokes equations and continuity in the form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6}$$

$$\rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \rho g_x \tag{7}$$

$$\rho \left[\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right] = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \rho g_y \tag{8}$$

The components of velocity can be written as $u = \overline{U} + \hat{u}$, $v = \hat{v}$, where $\hat{u} = \frac{\partial \Psi}{\partial y}$, $\hat{v} = -\frac{\partial \Psi}{\partial x}$, and considering only first order perturbations, we have, in dimensionless form, the equation:

$$\left(\frac{\partial}{\partial t} + \overline{U} \frac{\partial}{\partial x} \right) \nabla^2 \Psi - \frac{\partial^2 \overline{U}}{\partial y^2} \frac{\partial \Psi}{\partial x} = \frac{1}{Re} [\nabla^2 (\nabla^2 \Psi)] \tag{9}$$

Consider the normal mode in the dimensionless form $\Psi(x,y,t) = \hat{\Psi}(y) e^{i\alpha(x-ct)}$, where $\alpha = kh \in R$, $c = \frac{\omega}{k} \in C$, k is the wave number, therefore α is the characteristic wave number of the problem, and ω is the frequency. Using $\Psi(x,y,t)$ in Eq. (9) and making the contraction of notation for $D = \frac{\partial}{\partial y}$, we obtain the Orr-Sommerfeld Equation in dimensionless form, given by:

$$(D^2 - \alpha^2)^2 \hat{\Psi}(y) = i\alpha Re[(\overline{U} - c)(D^2 - \alpha^2) - D^2 \overline{U}] \hat{\Psi}(y) \tag{10}$$

2.2 Boundary Conditions.

Here we present the boundary conditions of the problem, which are: wall condition, kinematic condition of the interface and dynamic condition of the interface. In order to obtain these conditions, the same procedure used in the previous section is used here. The wall condition is given by:

$$u = 0 \ \ , y = -h \tag{11}$$

$$v = 0 \ \ , y = -h \tag{12}$$

$$D\hat{\Psi}(-1) = 0 \tag{13}$$

$$\hat{\Psi}(-1) = 0 \tag{14}$$

Figure 1 presents the interface $H(x, y, t)$ and the corresponding vectors. For the two-dimensional problem it is possible consider, $H(x, y, t) = y - \eta(x, t)$. The unit vectors normal and tangent to the interface, when linearised, are given by:

$$\vec{n} = -\frac{\partial \eta(x,t)}{\partial x}\vec{e_x} + \vec{e_y} \tag{15}$$

$$\vec{t} = \vec{e_x} + \frac{\partial \eta(x,t)}{\partial x}\vec{e_y} \tag{16}$$

where $\eta(x, t)$ and $\frac{\partial \eta(x,t)}{\partial x}$ are the position and inclination of the interface, respectively. The kinematic condition is,

$$\vec{u} \cdot \vec{n} = \vec{w} \cdot \vec{n} \ \ em \ \ y = \eta(x, t) \tag{17}$$

where,

$$\vec{u} \cdot \vec{n} = -u\frac{\partial \eta(x,t)}{\partial x} + v \tag{18}$$

$$\vec{w} \cdot \vec{n} = \frac{\partial \eta(x,t)}{\partial t} \tag{19}$$

Replacing Eq. (18), Eq. (19) in Eq. (17), using the normal modes of $\Psi(x, y, t)$, $\eta(x, t) = \hat{\eta}e^{i\alpha(x-ct)}$, and linearizing the equation around $y = 0$,

$$\hat{\Psi}(0) - (c - 1)\hat{\eta} = 0 \tag{20}$$

The dynamic condition at the interface has two parts. Considering that the surface between liquid and gas has no surfactants or temperature gradient; the first condition is the continuity of the tangential stress, associated with the viscous stress of the fluid. The second one is the continuity of the normal stress associated with the surface tension. The stress in the fluid is given by $\Sigma \cdot \vec{n}$, where Σ is the stress tensor, and reducing the effect of air to a purely normal stress $-P_0\vec{n}$, the dynamic conditions will be given by:

$$\vec{t} \cdot (\Sigma \cdot \vec{n}) = 0 \tag{21}$$

$$P_1 - P_2 = -\gamma(\nabla \cdot \vec{n}) \Rightarrow \vec{n} \cdot (\Sigma \cdot \vec{n}) - \vec{n} \cdot (-P_0\vec{n}) = -\gamma(\nabla \cdot \vec{n}) \ \ ; \ \ y = \eta(x, t) \tag{22}$$

where γ is the surface tension. Inserting the normal modes in Eqs. (21) and (22) and linearizing around $y = 0$, we obtain:

$$D^2\hat{\Psi}(0) + \alpha^2\hat{\Psi}(0) + \hat{\eta}D^2\overline{U}(0) = 0 \tag{23}$$

$$-D^3\hat{\Psi}(0) + [3\alpha^2 - i\alpha Re(c - 1)]D\hat{\Psi}(0) + i\alpha Re\left[\frac{1}{Fr} + \frac{\alpha^2}{We}\right]\hat{\eta} = 0 \tag{24}$$

3. Asymptotic Solution.

This section is devoted to the solutions of the equation of Orr-Sommerfeld, together with the boundary conditions of the problem, by means of an asymptotic analysis considered by Chimetta and Franklin (2015). To find these solutions we will expand the eigenfunction $\hat{\Psi}(y)$ and the eigenvalue c in power series of α, from $O(1)$ to $O(\alpha^2)$.

3.1 Solution for $O(1)$.

For a long wave disturbance, the wave number α can be treated as a small parameter. The equations suggest the speed c and the amplitude $\hat{\Psi}$ of eigenfunctions can be treated as a power series of α. In order to achieve an approximate solution we will replace the expansions into the Orr-Sommerfeld equation and the terms of the same order will be collected. The very same procedure will be applied on the boundary conditions.

At $O(1)$:

$$D^4\hat{\Psi}_0(y) = 0 \tag{25}$$

$$\hat{\Psi}_0(-1) = 0 \tag{26}$$

$$D\hat{\Psi}_0(-1) = 0 \tag{27}$$

$$\hat{\Psi}_0(0) - (c_0 - 1)\hat{\eta} = 0 \tag{28}$$

$$D^3\hat{\Psi}_0(0) = 0 \tag{29}$$

$$D^2\hat{\Psi}_0(0) - 2\frac{\hat{\Psi}_0(0)}{c_0 - 1} = 0 \tag{30}$$

From equations (25) to (30),

$$\hat{\Psi}_0(y) = \hat{\eta}(y+1)^2; \;\; c_0 = 2 \tag{31}$$

where $\hat{\eta}$ is the amplitude of deformation of the interface. The eigenvalue c_0 is real and independent of wave number; therefore, all disturbances are propagated with the same speed $2U_0$, independent of the wave (non-dispersive). Since the imaginary part of c_0 is zero, the growth rate of instability is zero, so there is no instability at $O(1)$.

3.2 Solution for $O(\alpha)$.

At $O(\alpha)$:

$$D^4\hat{\Psi}_1(y) = 4iRe\hat{\eta}y \tag{32}$$

$$\hat{\Psi}_1(-1) = 0 \tag{33}$$

$$D\hat{\Psi}_1(-1) = 0 \tag{34}$$

$$\hat{\Psi}_1(0) = c_1\hat{\eta} \tag{35}$$

$$D^2\hat{\Psi}_1(0) = 0 \tag{36}$$

$$D^3\hat{\Psi}_1(0) = -2iRe\hat{\eta} + iRe\hat{\eta}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right) \tag{37}$$

With these equations we find:

$$\hat{\Psi}_1(y) = iRe\hat{\eta}\left\{\frac{y^5}{30} + \left[-\frac{1}{3} + \frac{1}{6}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right]y^3 + \left[\frac{5}{6} - \frac{1}{2}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right]y + \frac{8}{15}\left[1 - \frac{5}{8}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right]\right\} \tag{38}$$

$$c_1 = iRe\frac{8}{15}\left[1 - \frac{5}{8}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right] \tag{39}$$

At $O(1)$ the solution is purely imaginary. We found $\sigma = \alpha c_i = \alpha^2 c_{1i}$ in the form,

$$\sigma = \frac{\alpha^2 Re}{3}\left(\frac{1}{Fr_c} - \frac{1}{Fr}\right) - \frac{Re}{3We}\alpha^4 \;\; \text{where} \;\; Fr_c = \tfrac{5}{8} \tag{40}$$

When $Fr < Fr_c$, σ is negative for every α, and the flow of the liquid film is linearly stable. For $Fr > Fr_c$, perturbations of wave number below α_c will be amplified. We can find α_c by,

$$\sigma = 0 \Leftrightarrow \alpha^2\left[\frac{Re}{3}\left(\frac{1}{Fr_c} - \frac{1}{Fr}\right) - \frac{Re}{3We}\alpha^2\right] = 0 \tag{41}$$

Excluding the case $\alpha^2 = 0$ we have,

$$\alpha_c^2 = We\left(\frac{1}{Fr_c} - \frac{1}{Fr}\right) \text{ with } Fr_c = \tfrac{5}{8} \tag{42}$$

Perturbations with wave number $\alpha > \alpha_c$ are attenuated due to the combined effect of surface tension and viscosity. The number $Fr_c = \frac{5}{8}$ is the critical Froude number above which the liquid film is unstable. For a limiting case of Froude number to the Eq. (42) we have,

$$\lim_{Fr\to\infty} \alpha_c^2 = \lim_{Fr\to\infty} \sqrt{We\left(\frac{1}{Fr_c} - \frac{1}{Fr}\right)} \to \sqrt{\frac{We}{Fr_c}} \tag{43}$$

where the Eq. (43) it sets that the result of the unstable band should be limited for a Froude number large enough.

3.3 Solution for $O(\alpha^2)$.

At $O(\alpha^2)$:

$$D^4\hat{\Psi}_2(y) = 4\hat{\eta} - Re^2\hat{\eta}\left\{-\frac{3}{5}y^5 + \left[\frac{2}{3} - \frac{2}{3}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right]y^3 + \left[\frac{11}{3} - 2\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right]y\right\} \tag{44}$$

$$\hat{\Psi}_2(-1) = 0 \tag{45}$$

$$D\hat{\Psi}_2(-1) = 0 \tag{46}$$

$$\hat{\Psi}_2(0) = \hat{\eta}c_2 \tag{47}$$

$$D^2\hat{\Psi}_2(0) = -\hat{\eta} \tag{48}$$

$$D^3\hat{\Psi}_2(0) = 6\hat{\eta} + Re^2\hat{\eta}\left[\frac{121}{30} - \frac{5}{2}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right] \tag{49}$$

And it is possible to obtain,

$$\hat{\Psi}_2(y) = \frac{\hat{\eta}}{6}y^4 - \frac{Re^2\hat{\eta}}{60}\left\{-\frac{1}{84}y^9 + \frac{1}{21}\left[1 - \left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right]y^7 + \left[\frac{11}{6} - \left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right]y^5\right\} + \frac{A}{6}y^3 + \frac{B}{2}y^2 + Cy + D \tag{50}$$

$$c_2 = -2 - \frac{128}{105}Re^2\left[1 - \frac{5}{8}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right] \tag{51}$$

where A, B, C and D are, respectively,

$$A = 6\hat{\eta} + Re^2\hat{\eta}\left[\frac{121}{30} - \frac{5}{2}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right] \tag{52}$$

$$B = -\hat{\eta} \tag{53}$$

$$C = -\frac{10\hat{\eta}}{3} - Re^2\hat{\eta}\left[\frac{625}{336} - \frac{209}{180}\left(\frac{1}{Fr} + \frac{\alpha^2}{We}\right)\right] \tag{54}$$

$$D = \hat{\eta}c_2 \tag{55}$$

This correction in c only affects the wave speed, the real part of eigenvalue, therefore, at $O(\alpha^2)$, long wavelengths are weakly dispersive (Benney, 1966).

4. Numerical Solution.

For the numerical solution we implemented a Galerkin method using Chebyshev polynomials for the discretization, in particular we use the Chebyshev polynomials of the first kind, known as T_n. Before implement the method it is necessary transfer the problem domain to the interval $[-1;1]$, this is necessary in order to use the Chebyshev polynomials once they are orthogonal in this interval. We use the transformation,

$$z = 2y + 1; \;\; for \;\; y \in [-1;0]. \tag{56}$$

after applying the transformation (56) and rearranging the boundary conditions in order to eliminate the term $\hat{\eta}$, we can discretize $\hat{\Psi}(z)$ with the approximation,

$$\hat{\Psi}(z) = \sum_{k=0}^{N} a_k T_k(z); \;\; k \in \{Z \mid k \geq 0\} \tag{57}$$

in this way, it is possible write the Orr-Sommerfeld equation in terms of the inner products required in the Galerkin method. After this process, we can write the problem as eigenvalue problem in the form,

$$[\mathbf{A}]_{NxN}\vec{a} = c[\mathbf{B}]_{NxN}\vec{a} \tag{58}$$

where N is the number of Chebyshev polynomials to be used and the matrices **A** and **B** can be respectively written as $\mathbf{A} = A_r + iA_i$ and $\mathbf{B} = B_r + iB_i$. To apply the boundary conditions we use the same approximation (57) for $z = -1$ for wall conditions and $z = 1$ for the interface conditions, and then replace the last four lines in the system (58) with the transform boundary conditions. Those choices were made based on the particular aspects of the problem. The Galerkin method was choose because of the complexity of the boundary conditions at the interface once, in the Galerkin method, we can use each equation of the boundary conditions as line vectors in the final matrices. The choice of using Chebyshev polynomials was made because of their high accuracy, and their orthogonal properties, which makes the implementation easier. A code was written in the MATLAB environment to solve the Eq. (58). We used the function 'eig' which uses a Cholesky factorization or a generalized Schur decomposition (QZ algorithm) based on the properties of **A** and **B**. If **A** and **B** are symmetric, the standard choice will be the Cholesky factorization, otherwise the software will implement the QZ algorithm. All codes implemented in this work, using the Galerkin formulation, were made using a number of Chebyshev polynomials equal 80.

5. Discussion and Results.

As presented in section 3, no instabilities appear at $O(1)$. Since the ratio of amplitudes $\hat{\eta}$ and $\hat{\Psi}_0$ are real, the interface η and the stream function Ψ are in phase. The perturbation velocities, $u = \frac{\partial \Psi}{\partial y}$ and $v = -\frac{\partial \Psi}{\partial x}$ are, respectively, in and out of phase with the interface. For smaller orders the flow does not manifest any instability for long-wave disturbances, (Charru, 2011). At $O(\alpha)$, we found the correction of eigenvalue $c_1 = iRe\frac{8}{15}[1 - \frac{5}{8}(\frac{1}{Fr} + \frac{\alpha^2}{We})]$, which affects the growth rate of instability. This correction generates as a result a critical Froude number equal to $\frac{5}{8}$, which, from Eq. (5), provides the critical Reynolds number $Re_c = \frac{5}{4}cot(\theta)$; therefore, the same condition discussed at the end of section (3.2) can be applied for the critical Reynolds number. This results agrees with Benney's results (Benney, 1966) for the $O(\alpha)$, which is given by $c_1 = iRe(Re - \frac{5}{4}cot(\theta)) = iRe^2(1 - \frac{5}{8}\frac{1}{Fr})$. Benney disregarded the contributions of Weber number until $O(\alpha^3)$, however the same criteria for the onset of instabilities was obtained. The solution for $O(\alpha^2)$ is given by Eq. (51) which affects the phase velocity. The value found by Benney was $c_2 = -2 - \frac{32}{63}Re(Re - \frac{5}{4}cot(\theta)) = -2 - \frac{32}{63}Re^2(1 - \frac{5}{8}\frac{1}{Fr})$ which shows a difference of 2.4, approximately, in the multiplicative constant. This difference is due to the contribution of $\frac{\alpha^2}{We}$ terms that are associated with the surface tension.

We have plot the growth rate $\sigma(\alpha)$ (see Fig. 2), and the stability diagram $\alpha(Fr)$ (see Fig. 5), given by Eq. (40) and Eq. (42). We use the physical properties $\mu = 0.001Ns/m^2$, $\rho = 998.2071Kg/m^3$, $g = 10m/s^2$, $\gamma = 0.07275N/m$ for temperature $T = 20^oC$ and 0.1 mm thickness, as reference in these plots. For the surface tension γ we use the work of Vargaftik *et al.* (1983) as a reference. In Fig. (2) it was used $\frac{\pi}{8} < \theta < \frac{\pi}{5.8}$ in order to obtain the $Fr < Fr_c$ and $Fr > Fr_c$ with the asymptotic solution. Figure 3 was made using the numerical approach, for the same range of θ, showing that asymptotic and numerical solution matches with high accuracy for the growth rate. Figure 4 was made with the numerical data and presents the stability diagram with the marginal stability curve ($\sigma = 0$, and $0 < \theta < \frac{\pi}{2.5}$), which separates the stable and unstable domains. Both domains are represent by lines corresponding to negative and positive values for the growth rate, the negative (at left and above the zero curve) and positive (at right and below the zero curve) values correspond to stable and unstable regions respectively. This diagram shows that the width of the unstable band tends to zero at the threshold $Fr = Fr_c$ and also that we have different behaviours for the growth rate according with the range of α. For the interval $0.04 < \alpha < 0.05$ we can see that the values for the growth rate present in the lines increase "faster"compared to other intervals, implying that this range for the wavenumber is more affected by an increment in the wall slope. Figure 5 shows a comparison between the asymptotic and numerical solutions for the marginal stability curve, both are in good agreement, specially below Froude number equals 1. However, above this value both lines separate Eq. (43) stablish that a limit for the wave number should be $\alpha_c = 0.0705$, based on the physical properties, and as we can see both solutions are in good agreement with this limit, so even for a higher Froude number a small error will be present between both results.

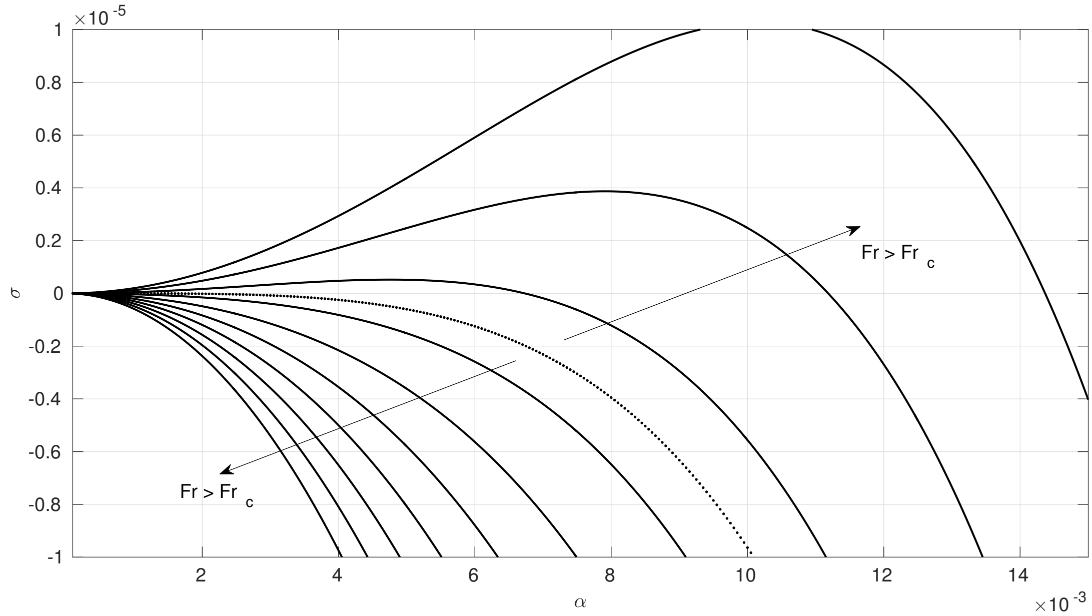

Figura 2: Behavior of the growth rate $\sigma(\alpha)$ for $Fr < Fr_c$ and $Fr > Fr_c$ with the asymptotic solution. The dotted line represents the growth rate for the critical Froude number.

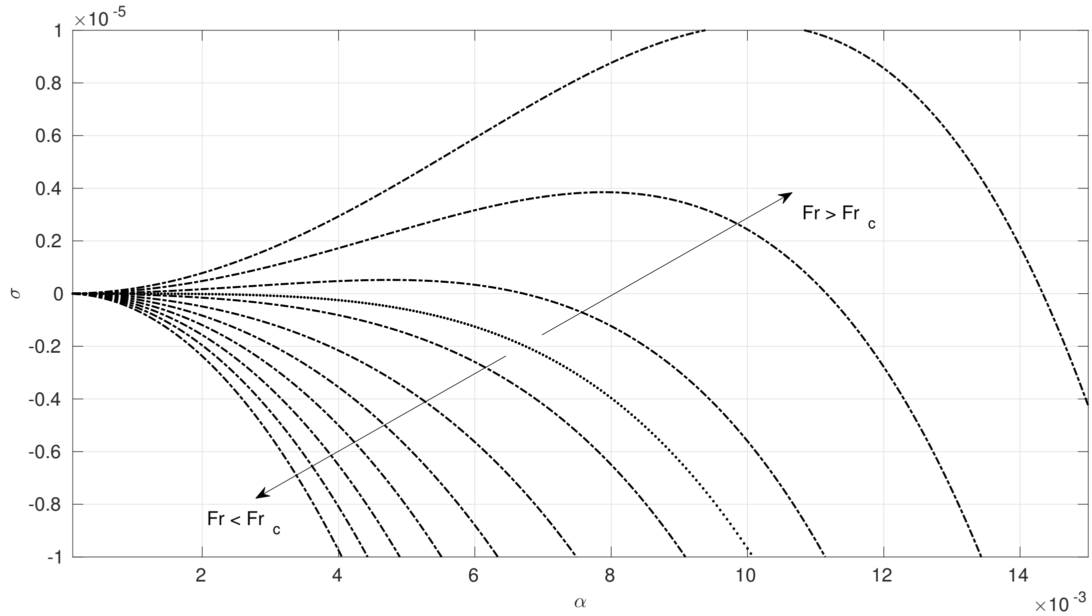

Figura 3: Behavior of the growth rate $\sigma(\alpha)$ for $Fr < Fr_c$ and $Fr > Fr_c$ with the numerical data. The dotted line represents the growth rate for the critical Froude number.

6. Conclusion.

The asymptotic analysis is a useful method to provide a good physical sense of the problem. Performing the analysis at $O(1)$ to $O(\alpha)$ we found the celerity and the growth rate of instability, both in good agreement with Benney (1966). At the $O(\alpha^2)$, our result for c_2 differs from Benney's result by a factor 2.4. This difference is due to the fact that in our resolution, we consider the contributions of $\frac{\alpha^2}{We}$ at $O(\alpha^2)$, while Benney only consider the contributions of Weber number starting from $O(\alpha^3)$. With the correction c_1 at $O(\alpha)$, it was possible to find the critical Froude number, which determines a critical condition between the effects of inertia and gravity: when $Fr < \frac{5}{8}$ the gravity effect dominates, and the liquid film is stable, and when $Fr > \frac{5}{8}$ the inertial effects dominates the flow and the liquid film become unstable. In possession of the critical Froude number we developed an expression for the instability curves, as well as the marginal stability diagram. With the Galerkin method it was possible to validate the asymptotic solution and find others structures for the growth rate besides the neutral curve ($\sigma = 0$) in the instability diagram. This method shows itself as a good alternative to implement complex boundary conditions in a easier way in order to solve stability problems.

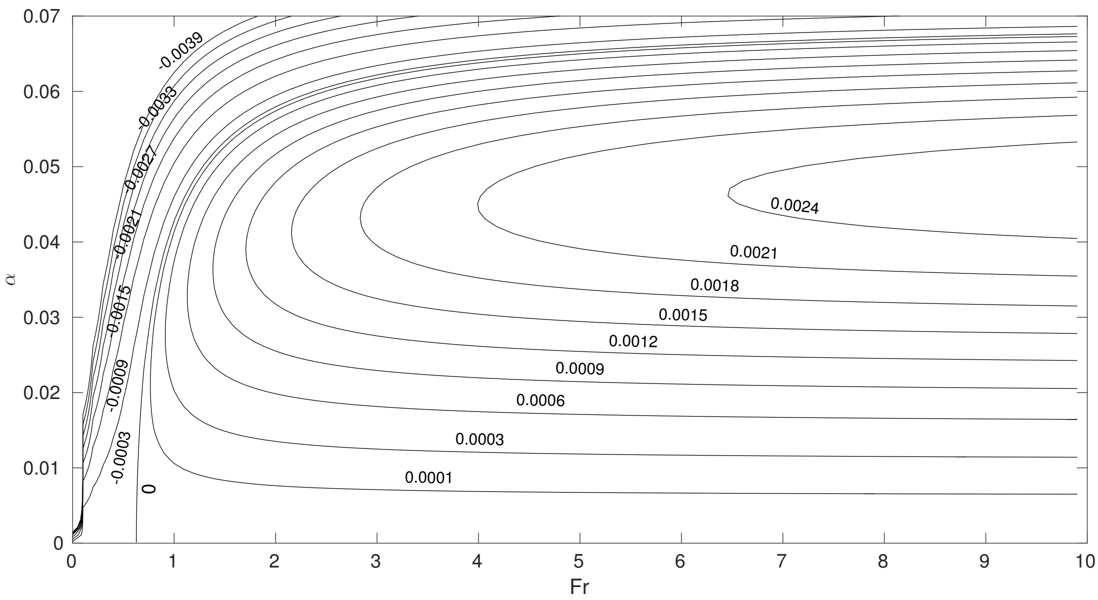

Figura 4: Stability diagram.

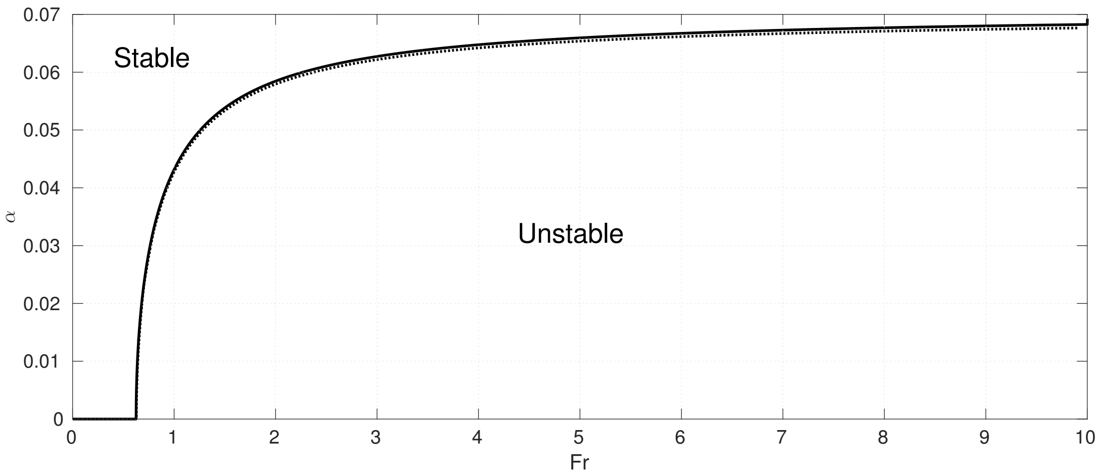

Figura 5: Comparison between the stability diagram for asymptotic and numerical solution. The continuous line represents Eq. (42) for the asymptotic solution and the dotted line is the numerical data.

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