24th ABCM International Congress of Mechanical Engineering
December 3-8, 2017, Curitiba, PR, Brazil

COBEM-2017-0516
HEAT CONDUCTION ANALYSIS IN LAYERED MEDIA WITH ADHESIVE DEFECT VIA SINGLE DOMAIN FORMULATION, INTEGRAL TRANSFORMS AND FINITE DIFFERENCE

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Abstract. *This work addresses the hybrid numerical-analytical solution of heat conduction in multilayered media by means of a single domain formulation, integral transforms and finite difference. The basic idea is to incorporate different materials into space variable coefficients with abrupt transitions, representing different subdomains of the original problem. The Classical Integral Transform Technique is then employed and the problem solution is written as an expansion in terms of eigenfunctions obtained from an eigenvalue problem with space variable coefficients. In order to solve this eigenvalue problem, the finite difference method with a second-order accurate scheme is employed. As illustration, the heat conduction problem in a multilayered pipe is considered, including the analysis of a defect due to the lack of adhesive between the layers. A purely numerical solution via the finite element method is also implemented with the Comsol Multiphysics.*

Keywords: *Single Domain Formulation, Integral Transforms, Layered Media, Finite Difference Method, Hybrid Methods.*

1. INTRODUCTION

The analysis of heat conduction in layered media appears in several contexts such as thermal insulation, corrosion protection, and layered composites, which have been providing increasing opportunities for tailoring structures to meet different requirements in modern construction materials (Naveira-Cotta et. al, 2009; Grosso et. at, 2016). The present work is aimed at extending the methodology that combines the single domain formulation and integral transforms (Cotta et. al, 2016) for the solution of heat conduction problems in layered media formulated in a single domain. The main contribution of this work consists in proposing a hybrid solution that combines the Classical Integral Transform Technique (Mikhailov and Özisik, 1984) with the Finite Difference Method (Pletcher and Anderson, 1997). The first is employed as the main solution method, while the latter is employed in the solution of the corresponding eigenvalue problem with space variable coefficients.

As illustration, the heat conduction problem in a multilayered pipe is considered, including the analysis of a defect due to the lack of adhesive between the layers. A purely numerical solution via the finite element method is also implemented with the Comsol Multiphysics CFD solver in order to allow for critical comparisons and coverification.

2. PROBLEM FORMULATION AND SOLUTION

Consider the analysis of a small region in a cylindrical pipe's wall, in such a way the problem can be simplified in Cartesian coordinates, as shown in Figure 1. It is considered the pipe is coated with a thermal insulating layer and there is a small defect due to lack of adhesive, which is represented by a region of air instead of adhesive, for instance due to the formation of a bubble during the application. In the example here considered, a heat flux q" is prescribed at the inner surface of the pipe, while the external surface exchanges heat by convection with the surrouding environment at temperature T_∞, and considering a heat transfer coeffficient h.

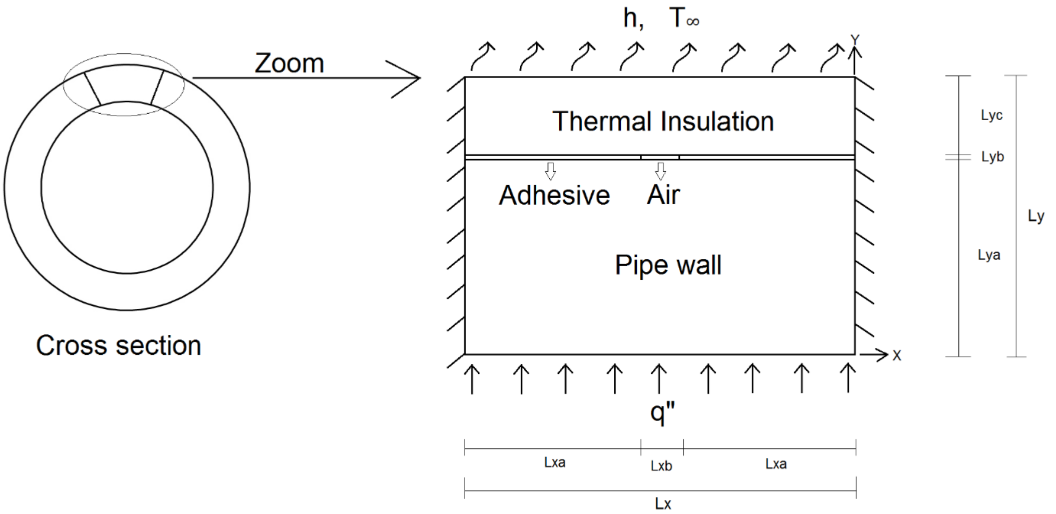


Figure 1. Schematic representation of the problem

This problem can be written in a single domain formulation as follows:

$$\rho(x,y)c_p(x,y)\frac{\partial T(x,y,t)}{\partial t}=\frac{\partial}{\partial x}\left(k(x,y)\frac{\partial T(x,y,t)}{\partial x}\right)+\frac{\partial}{\partial y}\left(k(x,y)\frac{\partial T(x,y,t)}{\partial y}\right) \tag{1}$$

$$\left.\frac{\partial T(x,y,t)}{\partial x}\right|_{x=0}=0 \qquad \left.\frac{\partial T(x,y,t)}{\partial x}\right|_{x=L_x}=0 \tag{2}$$

$$-k(x,y)\left.\frac{\partial T(x,y,t)}{\partial y}\right|_{y=0}=q'' \qquad k(x,y)\left.\frac{\partial T(x,y,t)}{\partial y}\right|_{y=L_y}+hT(x,y,t)\big|_{y=L_y}=hT_\infty \tag{3}$$

$$T(x,y,t)\big|_{t=0}=T_0 \tag{4}$$

where $\rho(x,y)$ is the specific mass, $c_p(x,y)$ is the specific heat, $k(x,y)$ is the thermal conductivity, all of them modeled as space variable functions in order to represent the properties of each material involved (pipe wall, adhesive, insulating layer and air) and presenting, therefore, abrupt transitions at these regions interfaces.

In order to homogenize the boundary condition at y = Ly, the following analytical filtering is employed:

$$T(x,y,t)=T_\infty+T^*(x,y,t) \tag{5}$$

in such a way that the following resulting problem is obtained:

$$\rho(x,y)c_p(x,y)\frac{\partial T^*(x,y,t)}{\partial t}=\frac{\partial}{\partial x}\left(k(x,y)\frac{\partial T^*(x,y,t)}{\partial x}\right)+\frac{\partial}{\partial y}\left(k(x,y)\frac{\partial T^*(x,y,t)}{\partial y}\right) \tag{6}$$

$$\left.\frac{\partial T^*(x,y,t)}{\partial x}\right|_{x=0}=0 \qquad \left.\frac{\partial T^*(x,y,t)}{\partial x}\right|_{x=L_x}=0 \tag{7}$$

$$-k(x,y)\left.\frac{\partial T^*(x,y,t)}{\partial y}\right|_{y=0}=q'' \qquad k(x,y)\left.\frac{\partial T^*(x,y,t)}{\partial y}\right|_{y=L_y}+hT^*(x,y,t)\Big|_{y=L_y}=0 \tag{8}$$

$$T^*(x,y,t)\Big|_{t=0}=T_0-T_\infty \tag{9}$$

Following the Classical Integral Transform Technique formalism (Mikhailov and Özisik, 1984), the integral transformation pair is proposed:

$$T^*(x,y,t)=\sum_{i=1}^{n}\tilde{\psi}_i(x,y)\overline{T}_i(t) \tag{10}$$

$$\overline{T}_i(t)=\int_0^{L_x}\int_0^{L_x}\rho(x,y)c_p(x,y)\tilde{\psi}_i(x,y)T^*(x,y,t)dxdy \tag{11}$$

where the eigenfunctions $\psi_i(x,y)$ are obtained by the solution of the following differential eigenvalue problem with variable coefficients:

$$\frac{\partial}{\partial x}\left(k(x,y)\frac{\partial\psi(x,y)}{\partial x}\right)+\frac{\partial}{\partial y}\left(k(x,y)\frac{\partial\psi(x,y)}{\partial y}\right)+\mu^2\rho(x,y)c_p(x,y)\psi\left(x,y\right)=0 \tag{12}$$

$$\left.\frac{\partial\psi(x,y)}{\partial x}\right|_{x=0}=0 \qquad \left.\frac{\partial\psi(x,y)}{\partial x}\right|_{x=L_x}=0 \tag{13}$$

$$\left.\frac{\partial\psi(x,y)}{\partial y}\right|_{y=0}=0 \qquad k(x,y)\left.\frac{\partial\psi(x,y)}{\partial y}\right|_{y=L_y}+h\psi(x,y)\Big|_{y=L_y}=0 \tag{14}$$

where

$$\tilde{\psi}_i(\boldsymbol{x})=\psi_i(\boldsymbol{x})/\sqrt{N_i} \tag{15}$$

Applying Eq. (11) in Eqs. (6) to (9), yields the transformed problem:

$$\frac{d\overline{T}_i(t)}{dt}+\mu_i^2\overline{T}_i(t)=\overline{g}_i(t) \tag{16}$$

where

$$\overline{g}_i(t)=-\int_S\frac{\phi(x,y,t)\left(k(x,y)\dfrac{\partial\tilde{\psi}_i(x,y)}{\partial \underset{\sim}{n}}-\tilde{\psi}_i(x,y)\right)}{\alpha(x,y)+\beta(x,y)}dS \tag{17}$$

The solution to the system of ordinary differential equations presented by Eq. (16) is:

$$\overline{T}_i(t)=e^{-\mu_i^2 t}\left[\overline{f}_i+\int_o^t e^{\mu_i^2 t}\overline{g}_i(t)dt\right] \tag{18}$$

where

$$\bar{f}_i = \int_0^{Lx}\int_0^{Ly} \rho(x,y)c_p(x,y)\tilde{\psi}(x,y)(T_0 - T_\infty)dydx \tag{19}$$

The most difficult task is the solution of the eigenvalue problem with space variable coefficients, Eqs. (12) to (14), which does not allow for a closed-form solution. The finite difference method, though a second-order accurate scheme, is employed in order to transform the differential eigenvalue problem into an algebraic eigenvalue problem, yielding:

$$\mathbf{A}\cdot\boldsymbol{\psi} + \mu^2\mathbf{B}\cdot\boldsymbol{\psi} = 0 \tag{20}$$

with

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_4 & 4\mathbf{I} & -\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \mathbf{0} & \cdots & \vdots \\ \mathbf{0} & \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I} & -4\mathbf{I} & -\mathbf{A}_4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}_1 \end{bmatrix} \tag{21.a,b}$$

where

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ c & b & f & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & c & b & f \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & e & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{21.c-e}$$

$$\mathbf{A}_4 = \begin{bmatrix} -3 & 4 & -1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{21.f,g}$$

and

$$a = k_{i-1/2,j}\Delta y^2,\ \ b = -\Delta y^2(k_{i+1/2,j} + k_{i-1/2,j}) - \Delta x^2(k_{i,j+1/2} + k_{i,j-1/2}),\ \ c = k_{i,j-1/2}\Delta x^2,$$
$$d = \rho_{i,j}c_{p\,i,j}\Delta x^2\Delta y^2,\ e = k_{i+1/2,j}\Delta y^2,\ f = k_{i,j+1/2}\Delta x^2. \tag{21.h-n}$$

Once the eigenvalues and corresponding eigenfunctions are known, the inversion formula, Eq. (10), can be readily employed in order to obtain the solution for the filtered problem, $T^*(x,y,t)$. The solution for the original potential, $T(x,y,t)$, can be readily retrieved through Eq. (5).

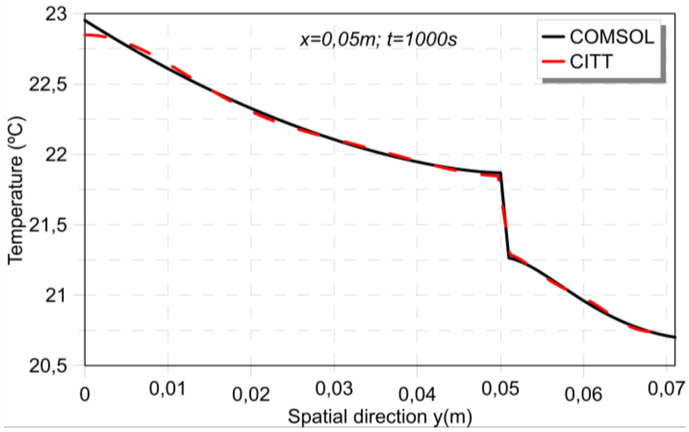
3. RESULTS AND DISCUSSION

In the example considered for the numerical results, the following parameters are considered: $Lya = 0.050\text{m}$, $Lyb = 0.001\text{m}$, $Lyc = 0.020\text{m}$, $Lxa = 0.045\text{m}$, $Lxb = 0.010\text{m}$, $k_{steel} = 13.4\text{W/mK}$, $k_{adhesive} = 0.7\text{W/mK}$,

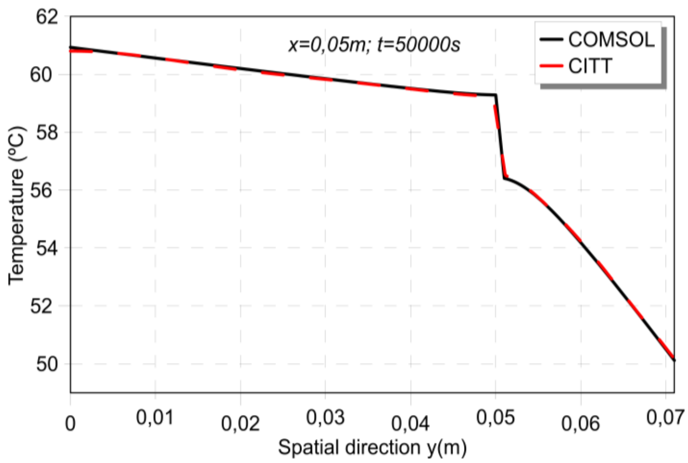
$k_{insulation} = 1.171 W/mK$, $k_{air} = 0.0263 W/mK$, $\rho cp_{steel} = 3.86 MJ/m^3K$, $\rho cp_{adhesive} = 1.75 MJ/m^3K$, $\rho cp_{insulation} = 2.65 MJ/m^3K$, $\rho cp_{air} = 1.17 J/m^3K$ (Incropera et. al, 2007; Grosso et. al, 2016), $q'' = 500 W/m^2$ e $T_0 = T_\infty = 20^o C$.

Figures 2(a-b) depict the temperature profiles along the *x*-axis at the position $x = 0.050m$ and at time instants $t = 1000s$ and $t = 50000s$, respectively, calculated with methodology here presented employing 100 terms in the eigenfunction expansion and a mesh with 101x101 nodes in finite difference solution of the eigenvalue problem. The results obtained with the Comsol Multiphysics CFD solver, employing the *Extremely Fine* mesh configuration are also presented, showing good adherence.

It can be noticed a slightly higher deviation between both solutions near y=0. This is caused by the non-homogeneous boundary condition at this position, Eq. (8), yielding slower convergence rates in the CITT (Classical Integral Transform Technique) solution.



a



b

Figure 2. Temperature profiles along the *x*-axis at $x = 0.050m$

Figure 3 depicts the temperature profiles along the *x*-axis at the position $y = 0.0505m$ and at time instant $t = 50000s$ calculated with CITT employing 100 terms in the eigenfunction expansion and a mesh with 101x101 nodes in finite difference solution of the eigenvalue problem.

The maximum deviation observed between CITT and Comsol solution is about only 0.025ºC, occurring at the defect region, where abrupt transitions at the thermal conductions are present.

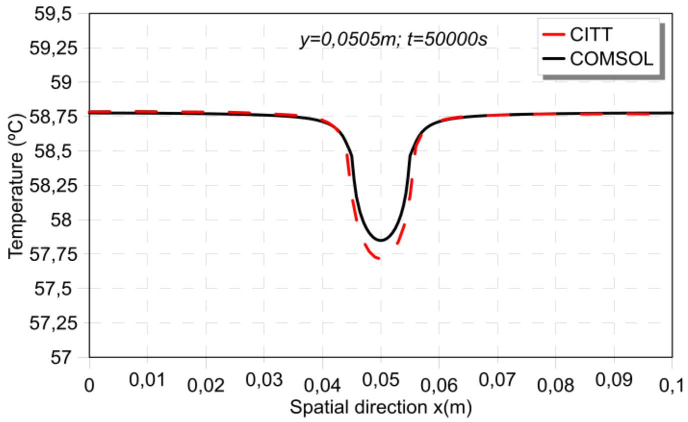


Figure 3. Temperature profiles along the *y*-axis at $y = 0.0505\text{m}$

The graphic in Figure 3 illustrate a good convergence for positions out of the defect between the hybrid solution using GITT and the numerical solution obtained by FDM. There is a temperature abrupt variation near the y=0,50m position because of the presence of air in this defect zone, where the solutions have a small divergence.

Tables 1(a-b) illustrate the convergence of the eigenfunction expansion, at $y = Ly$ and $y = 0.0505\text{m}$, respectively, at the time instant $t = 1000\text{s}$, for increasing truncation orders, employing a mesh with 301x301 nodes in the finite difference solution of the eigenvalue problem.

Table 1. Convergence of the eigenfunction expansion at (a) $y = 0.071\text{m}$; (b) $y = 0.0505\text{m}$.

(a)

Truncation Order	Temperature (ºC)	
	$x = 0.02\text{m}$	$x = 0.05\text{m}$
50	20.7513	20.6930
100	20.7552	20.6970
150	20.7600	20.7113
200	20.7599	20.6999
Comsol+	20.7590	20.7008

(b)

Truncation Order	Temperature (ºC)	
	$x = 0.02\text{m}$	$x = 0.05\text{m}$
50	21.7863	21.5984
100	21.7499	21.5726
150	21.7707	21.5862
200	21.7563	21.5784
Comsol+	21.7545	21.5673

+ Solution obtained with automatically generated mesh with “Extremely Fine” option.

The results obtained by CITT presented a convergence of at least three significant digits for 150 terms less. The CITT are also in agreement with the Comsol results with three significant digites.

4. CONCLUSIONS

The hybrid numerical-analytical solution employed in this work, combining single domain formulation, integral transforms and finite difference, has been demonstrated feasible for the solution of the heat conduction problem in layered media here considered as example. The research now continues towards the inverse analysis for the identification of defects in layered pipes employing external temperature measurements and combining the solution methodology here advanced with Bayesian inference.

5. ACKNOWLEDGEMENTS

The authors acknowledge the financial support provided by the sponsoring agencies CNPq, CAPES and FAPERJ.

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