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ENRICHED MODIFIED LOCAL GREEN'S FUNCTION METHOD APPLIED TO ELASTO STATIC PROBLEMS

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Abstract. *On one hand, we have a powerful integral technique called Modified Local Green's Function Method (MLGFM) that has been adopted to solve several problems in continuum mechanics last decades. It uses the finite element method (FEM) as an auxiliary tool to automatically develop the Green's functions projections without knowing the fundamental solutions in opposite of boundary element method (BEM). On the other hand, procedures based on the addition of new shape functions in conventional FEM known by Enriched Methods have been explored in order to improve its solution. In this paper, the Modified Local Green's Function Method is now enriched by Lobatto shape functions in the boundary, in the domain and in the both spaces. Some standard set of elasto static problems are employed and compared with the MLFGM, conventional FEM and analytical solution.*

Keywords: *Modified Local Green's Function Method. Enriched Methods. Hierarchical Finite Element. Lobatto Shape Functions.*

1. INTRODUCTION

The Modified Local Green's Function Method (MLGFM) is an integral technique that explores the main benefits from three other techniques: the Finite Element Method (FEM), the Boundary Element Method (BEM) and the Green's Function. The method was first proposed by Barcellos and Silva (Barcellos & Silva, 1987), which applied the MLGFM to the case of elastic membranes problems. In opposite of the BEM, the MLGFM does not use a fundamental solution. The matrices of the integral equations systems are determined directly without the explicit knowledge of the Green Function. The MLGFM uses finite elements at the domain to create discrete projections of the Green's Functions, which correspond to the fundamental solutions to be used in the integral equations system associated with the boundary approximation. When FEM and BEM are associated, the Modified Local Green's Function Method became as efficient as the former ones, making possible to apply it at any Continuum Mechanics problem, with any geometry and boundary condition (Barbieri and Barcellos, 1991; Barcellos and Barbieri, 1991; Machado and Barcellos, 1992; Maldaner and Barcellos, 1992; Filippin et al, 1992a, b, c; Barbieri et al, 1993; Machado et al, 2008; Machado et al, 2012).

The Finite Element Method (FEM) (Oden and Reddy, 1976; Becker et al., 1981; Bathe, 1996; Zienkiewicz and Taylor, 2000) is proven one of the most widespread techniques used for the approximate solution of partial differential equations. However, several authors observed that low polynomial order of the finite elements may give poor results for some applications and improvements have been proposed (Ribeiro, 2001; Arndt et al, 2010; Torii and Machado, 2012). Most improvements in FEM involve the enrichment of the approximation space by some set of functions such as seen in the hierarchical formulations (HFEM) and the Partition of Unity (PU) methods (Bardell, 1991; Houmat, 1997; Torii et al, 2015; Arndt et al, 2016; Shang et al, 2016). In the hierarchical enrichment case, the approximation space is populated by a set of shape functions with degree " $p+1$ " where the space of degree " p " is a sub-space of degree " $p+1$ ". In other words, the approximation space with degree " p " is "enriched" by shape functions of space with degree " $p+1$ ".

One example of polynomials used to enrich an approximation space in terms of hierarchical basis is the Lobatto shape functions (Solín et al, 2004).

In this paper, we present the first attempt to enrich the MLFGM approximation space using hierarchical Lobatto shape functions as described by Solín (Solín et al, 2004). Some elasto static problems to test the method accuracy are presented and compared with the MLFGM, the classic FEM and the analytical solution (MacNeal and Harder, 1985).

2. METHODOLOGY

2.1 The modified local Green's function method

The modified local Green's function method was developed by Barcellos and Silva (Barcellos & Silva, 1987) to solve continuum mechanics problems. Many researches have worked with this method, which has shown good results in elasticity, composite laminated plates, nonlinear fields, and thermal problems as mentioned in this text earlier.

The MLFGM is used to solve boundary value problems, supposing that Ω is any domain limited by boundaries $\Gamma_{\mathcal{D}}$ and $\Gamma_{\mathcal{N}}$ which are related to Dirichlet and Neumann boundary conditions respectively, such as $\Gamma_{\mathcal{D}} \cup \Gamma_{\mathcal{N}} = \partial \Omega$. The MLFGM solves any problem by two steps, the first one is the real problem, which is governed by the differential operators \mathcal{L} , \mathcal{D} , \mathcal{N} , and the second one is an auxiliary or adjoint problem, which is governed by the differential adjoint operators \mathcal{L}^* and \mathcal{N}^* . The two steps are drafted in the Tab. 1, where $\mathbf{u}(P)$ is the unknown variables, $\mathbf{a}(P)$ is the independent terms, $\mathbf{G}(P,Q)$ is the Green's function matrix, $\delta(P,Q)$ is the Dirac delta generalized function, \mathbf{I} is the identity operator, $\mathcal{N}^\#$ is an additional operator, P and Q are the domain source point and domain field point respectively, and p and q are the correspondent points on the boundary.

A good choice for the additional operator is such that it satisfies $\mathcal{N}^\# \mathbf{u}(p) = 0$ on $\Gamma_{\mathcal{N}}$. For the auxiliary adjoint problem, the Green's tensor $\mathbf{G}(P,Q)$ must be understood as the generalized displacement of point P in the direction of an unitary vector \mathbf{n}_P when a generalized force is applied on point Q , in the direction of an unitary vector \mathbf{n}_Q .

Table 1. The real and the auxiliary problems

Main (real) problem	Auxiliary (adjoint) problem
Find $\mathbf{u}(P)$ such that	Find $\mathbf{G}(P,Q)$ such that
$\mathcal{L}\mathbf{u}(P) = \mathbf{a}(P); \quad P \in \Omega \quad (a)$	$\mathcal{L}^* \mathbf{G}(P,Q) = \delta(P,Q)\mathbf{I}; \quad P, Q \in \Omega \quad (b)$
Subjected to	Subjected to
$\mathcal{D}\mathbf{u}(p) = \mathbf{b}(p); \quad p \in \Gamma_{\mathcal{D}} \quad (c)$	$(\mathcal{N}^* + \mathcal{N}^\#) \mathbf{G}(p,Q) = 0 \quad (d)$
$\mathcal{N}\mathbf{u}(p) = \mathbf{c}(p); \quad p \in \Gamma_{\mathcal{N}}$	

Some algebraic manipulations are necessary to determine the main expressions of the MLFGM. The reader is kindly referred to go through our reference (Barbieri et al, 1998a, b) for further details on the discretization procedure. It must be emphasized that the direct treatment of the Neumann operator may cause numerical troubles. This drawback can be avoided by introducing a vector $\mathbf{f}(p)$, associated to the boundary fluxes, and given by:

$$\mathbf{f}(p) = (\mathcal{N}^* + \mathcal{N}^\#)\mathbf{u}(p) \quad (1)$$

In such a way, one is possible to write:

$$\mathbf{u}(Q) = \int_{\Omega} [\mathbf{G}^T(P,Q)\mathbf{a}(P)]d\Omega + \int_{\Gamma} [\mathbf{G}^T(p,Q)\mathbf{f}(p)]d\Gamma; \quad P, Q \in \Omega; \quad p \in \Gamma; \quad (2)$$

To extend the integral equation into the boundary, the trace operator (Oden and Reddy, 1976) is applied:

$$\mathbf{u}(q) = \int_{\Omega} [\mathbf{G}^T(P,q)\mathbf{a}(P)]d\Omega + \int_{\Gamma} [\mathbf{G}^T(p,q)\mathbf{f}(p)]d\Gamma; \quad P \in \Omega; \quad p, q \in \Gamma; \quad (3)$$

Equations (2) and (3) describe completely the problem. Since these equations involve domain and boundary integrals, two types of meshes are necessary, one in the domain and the other on the boundary, using FEM and BEM methods, respectively. The FEM domain approximation is also used to develop the Green's functions which are associated to the matrices $\mathbf{G}(P,Q)$, $\mathbf{G}(p,Q)$, $\mathbf{G}(P,q)$ and $\mathbf{G}(p,q)$.

The basis shape functions are the same as in the conventional FEM and BEM methods. For the present work, the bilinear quadrilateral element is enriched by new shape functions in the domain and the linear element is enriched by a new shape function on the boundary.

By developing discrete equations from the nodal values, one obtains:

$$\mathbf{A}\mathbf{u}_\Omega = \mathbf{B}\mathbf{f} + \mathbf{C}\mathbf{a} \quad (\text{in the domain}) \quad (4)$$

$$\mathbf{D}\mathbf{u}_\Gamma = \mathbf{E}\mathbf{f} + \mathbf{F}\mathbf{a} \quad (\text{on the boundary}) \quad (5)$$

, where \mathbf{u}_Ω and \mathbf{u}_Γ are the domain and the boundary displacements, respectively, \mathbf{a} and \mathbf{f} are the independent and the fluxes variables vectors. The matrices \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{D} , \mathbf{E} and \mathbf{F} can be written as:

$$\mathbf{A} = \int_\Omega \boldsymbol{\psi}(Q)^T \boldsymbol{\psi}(Q) d\Omega ; \quad (6)$$

$$\mathbf{B} = \int_\Omega \boldsymbol{\psi}(Q)^T \mathbf{G}_\Gamma(Q) d\Omega ; \quad (7)$$

$$\mathbf{C} = \int_\Omega \boldsymbol{\psi}(Q)^T \mathbf{G}_\Omega(Q) d\Omega ; \quad (8)$$

$$\mathbf{D} = \int_\Gamma \boldsymbol{\phi}(q)^T \boldsymbol{\phi}(q) d\Gamma ; \quad (9)$$

$$\mathbf{E} = \int_\Gamma \boldsymbol{\phi}(q)^T \mathbf{G}_\Gamma(q) d\Gamma ; \quad (10)$$

$$\mathbf{F} = \int_\Gamma \boldsymbol{\phi}(q)^T \mathbf{G}_\Omega(q) d\Gamma ; \quad (11)$$

, where $\boldsymbol{\psi}(Q)$ and $\boldsymbol{\phi}(q)$ are matrices with the shape functions in the domain and on the boundary, respectively, and $\mathbf{G}_\Gamma(Q)$, $\mathbf{G}_\Omega(Q)$, $\mathbf{G}_\Gamma(q)$, $\mathbf{G}_\Omega(q)$ are the Green's function projections over the boundary Γ and the domain Ω , evaluated on the points Q and q . Note that $\boldsymbol{\phi}(q)$ must be the trace of $\boldsymbol{\psi}(Q)$. The Green's projections can be written as:

$$\mathbf{G}_\Gamma(Q) = \int_\Gamma \mathbf{G}^T(p, Q) \boldsymbol{\phi}(q) d\Gamma ; \quad (12)$$

$$\mathbf{G}_\Omega(Q) = \int_\Omega \mathbf{G}^T(P, Q) \boldsymbol{\psi}(P) d\Omega ; \quad (13)$$

$$\mathbf{G}_\Gamma(q) = \int_\Gamma \mathbf{G}^T(p, q) \boldsymbol{\phi}(q) d\Gamma ; \quad (14)$$

$$\mathbf{G}_\Omega(q) = \int_\Omega \mathbf{G}^T(P, q) \boldsymbol{\psi}(P) d\Omega ; \quad (15)$$

To determine the Green's functions automatically, one should consider the following functional \mathcal{F} , which depends on \mathbf{G}_Ω or \mathbf{G}_Γ :

$$\mathcal{F}(\mathbf{G}_\Omega, \mathbf{G}_\Gamma) = \mathcal{B}(G, G) - \alpha \mathcal{B}_1(\mathbf{G}_\Omega, \boldsymbol{\psi}) - \beta \mathcal{B}_2(\mathbf{G}_\Gamma, \boldsymbol{\phi}) + \mathcal{B}_3(\mathbf{G}, \mathbf{G}) ; \quad (16)$$

, where \mathbf{G} corresponds to \mathbf{G}_Ω or \mathbf{G}_Γ depends on the case of interest; \mathcal{B} is a bilinear form, developed to \mathbf{G}_Ω or \mathbf{G}_Γ ; α and β are constants whose values are: $\alpha = 1$ and $\beta = 0$ to determine \mathbf{G}_Ω , $\alpha = 0$ and $\beta = 1$ to determine \mathbf{G}_Γ ; and \mathcal{B}_1 , \mathcal{B}_2 and \mathcal{B}_3 are bilinear forms which can be written as:

$$\mathcal{B}_1(\mathbf{G}_\Omega, [\boldsymbol{\psi}]) = \int_\Omega \mathbf{G}_\Omega(Q) \boldsymbol{\psi}(Q) d\Omega ; \quad (17)$$

$$\mathcal{B}_2(\mathbf{G}_\Gamma, [\boldsymbol{\phi}]) = \int_\Gamma \mathbf{G}_\Gamma(q) \boldsymbol{\phi}(q) d\Gamma; \quad (18)$$

$$\mathcal{B}_3(\mathbf{G}, \mathbf{G}) = \frac{1}{2} \int_\Gamma \mathcal{N}^\#(q) \mathbf{G}(q) \cdot \mathbf{G}(q) d\Gamma; \quad (19)$$

The minimization of functional $\mathcal{F}(\mathbf{G}_\Omega, \mathbf{G}_\Gamma)$ in Eq. (16) results a linear equation system which can be solved to determine the Green's projections:

$$[\mathbf{K}][\mathbf{G}_\Omega(Q) \quad ; \quad \mathbf{G}_\Gamma(Q)] = [\mathbf{A} \quad ; \quad \mathbf{D}]; \quad (20)$$

, where $[\mathbf{K}]$ is the global stiffness matrix, evaluated as the same way of the conventional finite element stiffness matrix; \mathbf{A} and \mathbf{D} are the matrices of "Eq. (6)" and "Eq. (9)", respectively. In such a way, the Green's projections are determined directly from "Eq. (20)", and they can be applied in "Eq. (7)", "Eq. (8)", "Eq. (10)" and "Eq. (11)" to complete the matrices of "Eq. (4)" and "Eq. (5)", which are the main system of the MLGFM.

2.2 Enriched methods and polynomial HFEM

The main feature of the enriched methods is the enrichment of the shape functions space from the classical FEM by adding other polynomial or non-polynomial functions. The approximated solution of these methods – the finite element approximation of displacements - in the element domain, is obtained by:

$$\mathbf{u}_h^e = \mathbf{u}_{FEM} + \mathbf{u}_{ENRICHED} \quad (21)$$

, or in matrix form:

$$\mathbf{u}_h^e = \mathbf{N}_n^T \mathbf{d}_n + \mathbf{N}_e^T \mathbf{d}_e \quad (22)$$

, where \mathbf{u}_{FEM} is the finite element displacement field based on nodal degrees of freedom; $\mathbf{u}_{ENRICHED}$ the enriched displacement field based on field degrees of freedom; \mathbf{d}_n the conventional finite element degrees of freedom vector, the vector \mathbf{N}_n contains the classical finite element shape functions and the vectors \mathbf{N}_e and \mathbf{d}_e contain the enrichment functions and the field degrees of freedom, respectively. Different sets of enrichment functions can produce different enriched methods (Arndt et al, 2010; Torii et al, 2016).

In this work, we have considered 1D 2-node linear element on the boundary and 2D 4-node linear quadrilateral element in the domain for plane stress and plane strain problems. The hierarchical concept is developed using Lobatto shape functions (Solín et al. 2004). For example, the first five Lobatto shape functions are given by:

$$l_1 = \frac{1-\xi}{2}, \quad (23)$$

$$l_2 = \frac{1+\xi}{2}, \quad (24)$$

$$l_3 = \frac{1}{2} \sqrt{\frac{3}{2}} (\xi^2 - 1), \quad (25)$$

$$l_4 = \frac{1}{2} \sqrt{\frac{5}{2}} (\xi^2 - 1) \xi \quad (26)$$

And,

$$l_5 = \frac{1}{8} \sqrt{\frac{7}{2}} (\xi^2 - 1)(5\xi^2 - 1) \quad (27)$$

, where $\xi = [-1, 1]$ is the finite element natural coordinate, also known as local or parameterized coordinates. These shape functions are showed in "Fig. 1".

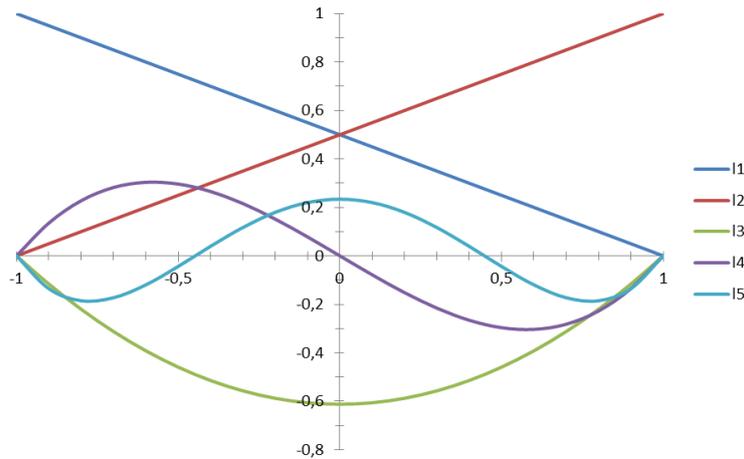


Figure 1. Five first Lobatto shape functions (Solín et al., 2004).

In the case of the 1D linear element shape functions on the boundary (“Fig. 2”), using the Lobatto shape functions, the resulting boundary shape functions $\phi_b(\xi)$ are:

$$\phi_b(\xi) = l_i(\xi), \forall i = 1, 2, \dots, n; \quad (28)$$

, where $\xi = [-1, 1]$ is the natural coordinate of a 1D finite element and n is the number of shape functions from Lobatto shape functions to be used.

Taking $n = 2$, for example, results in the following shape functions:

$$\phi_1(\xi) = l_1(\xi) = \frac{1-\xi}{2}; \quad (29)$$

$$\phi_2(\xi) = l_2(\xi) = \frac{1+\xi}{2}; \quad (30)$$

In the case of the 2D 4-node linear quadrilateral element shape functions in the domain (“Fig. 3”), we can multiply the Lobatto shape functions defined above for the one dimensional case. The resulting domain shape functions $\psi_d(\xi, \eta)$ are:

$$\psi_d(\xi, \eta) = l_i(\xi)l_j(\eta), \forall i, j = 1, 2, \dots, n; \quad (31)$$

, where $\xi = [-1, 1]$ and $\eta = [-1, 1]$ are the natural coordinates of a square master finite element and n is the number of shape functions from Lobatto shape functions to be used.

Taking $n = 2$, for example, results in the following shape functions:

$$\psi_1(\xi, \eta) = l_1(\xi)l_1(\eta) = \frac{1}{4}(1-\xi)(1-\eta) \quad (32)$$

$$\psi_2(\xi, \eta) = l_1(\xi)l_2(\eta) = \frac{1}{4}(1+\xi)(1-\eta) \quad (33)$$

$$\psi_3(\xi, \eta) = l_2(\xi)l_1(\eta) = \frac{1}{4}(1+\xi)(1+\eta) \quad (34)$$

And,

$$\psi_4(\xi, \eta) = l_2(\xi)l_2(\eta) = \frac{1}{4}(1-\xi)(1+\eta) \quad (35)$$



Figure 2. A 1D linear element with local coordinate $\xi = [-1, 1]$.

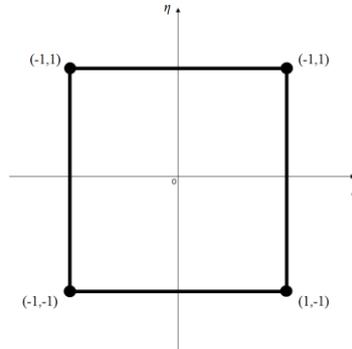


Figure 3. A 2D quadrilateral finite element with local coordinates $\xi = [-1, 1]$ and $\eta = [-1; 1]$.

Note that, for the above examples, the resulting shape functions are the ones commonly used in the 1D linear element and 2D bilinear quadrilateral element. The same procedure can be used to obtain the shape functions for an arbitrary value of n . Note that the degree of the approximation will be $p = n - 1$. The shape functions from “Eq. 32” to “Eq. 35” are called “vertex” shape functions by Solín (2004). They can be seen in “Fig. 4”. Other shape functions can be used to enrich the approximation for cases where $n \geq 2$. They are named as “edge” shape functions – related to the element boundary – and “bubble” shape functions – related to the element field (interior).

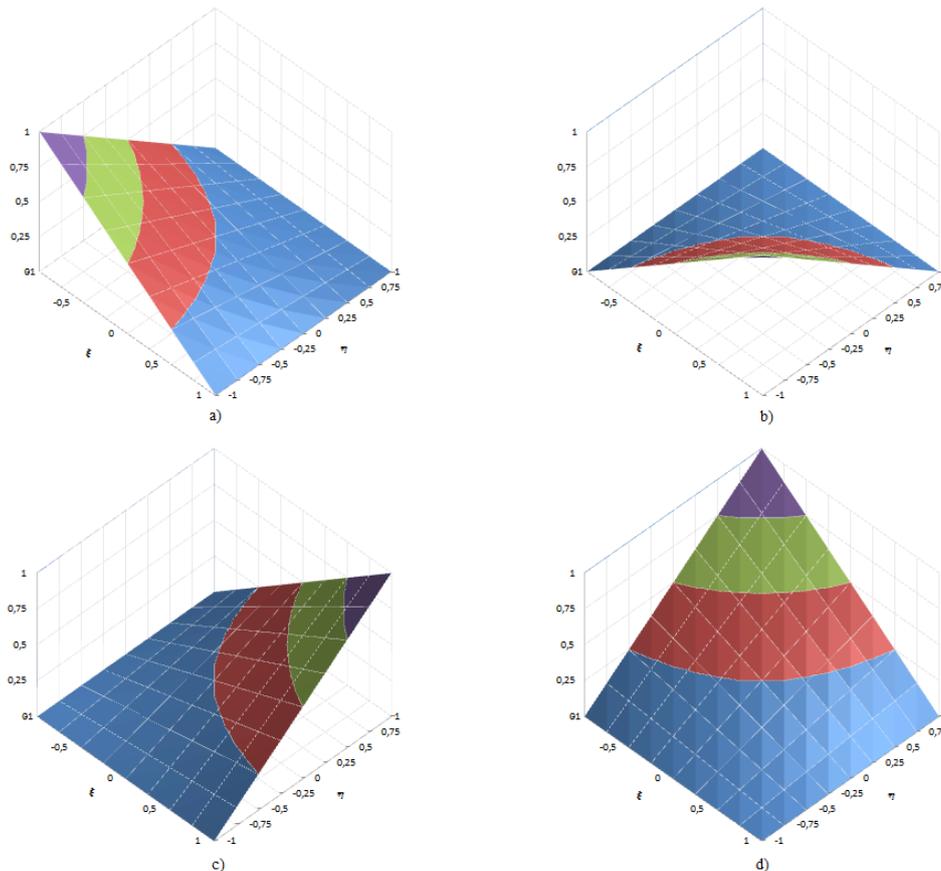


Figure 4. Shape functions for a quadrilateral finite element using $n = 2$.

3. RESULTS AND DISCUSSION

This section presents the first applications of enriched MLGFM using a 4-node bilinear quadrilateral element in the domain and a 2-node linear element on the boundary. A hierarchical enrichment methodology was adopted using Lobatto shape functions (Solín et al, 2004). The results from this enriched approach are compared to the MLGFM using a 4-node bilinear quadrilateral element in the domain and a 2-node linear element on the boundary, a 9-node biquadratic quadrilateral element in the domain and a 3-node quadratic element on the boundary, the classical FEM (for straight beam application only) and the analytical solution (MacNeal and Harder, 1985).

Next sessions are divided in order to include some set of problems proposed by Macneal and Harder (1985). First examples involve tensile and bending tests for straight beams in plane stress state. A plane strain state problem is also considered closing the results session.

3.1 Straight cantilever beam application

The straight cantilever beam cases are used in this session to test the Enriched Modified Local Green's Function Method (EMLGFM). The tensile and bending tests for beams in plane stress state proposed by MacNeal and Harder (1985) are taken as reference. The main goal of this problem is to verify the efficiency of the 2D elements relation to the distortion and aspect ratio. For this example, we have used an approach to enrich the domain and boundary simultaneously, only the domain and only the boundary (3 ways to enrich the MLGFM). Only one level of enrichment is used here ($p=2$). Their geometries, meshes and properties are showed in "Fig. 5".

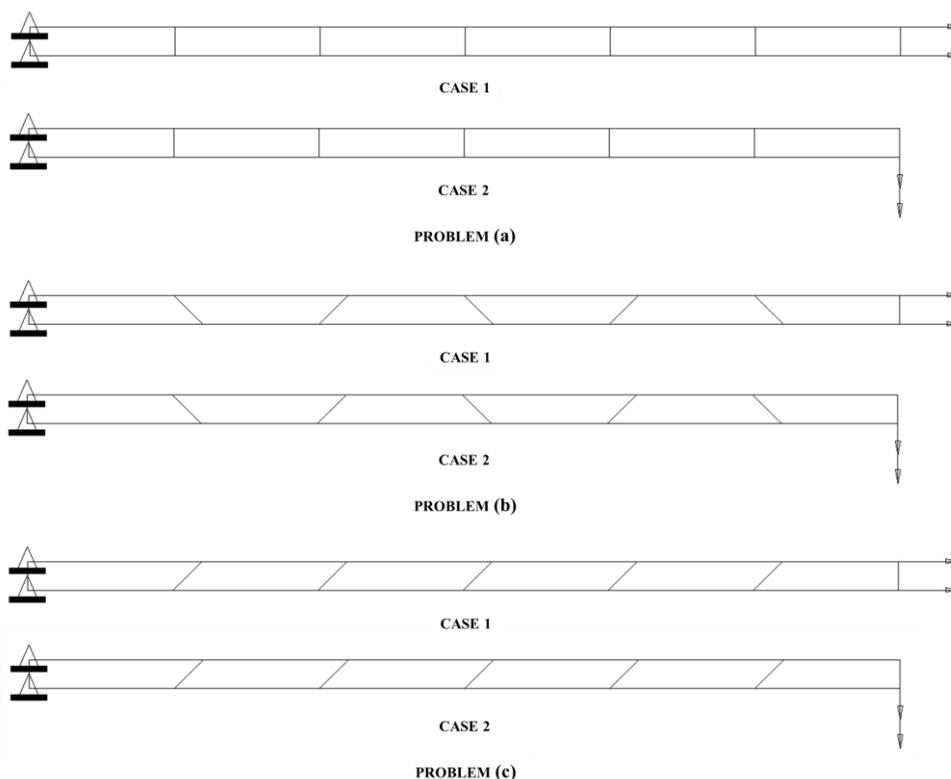


Figure 5. Straight cantilever beam: problem (a) - regular shape elements; problem (b) - trapezoidal shape elements; problem (c) - parallelogram shape elements. Length = 6.0; width (height) = 0.2; depth (thickness): 0.1; $E = 1.0 \times 10^7$; $\nu = 0.30$; mesh = 6×1 . Loading: unit forces at free end (tip). Constraint: fixed support.

As we can see in Tab. 2 (displacements results) for straight beam at tensile load (extension – case 1), all EMLGFM approaches presented good approximation compared to the theoretical solution even with irregular elements shape. Anyway, it's important to highlight the very low variation of the EMLGFM with domain/boundary enrichment. This approach has presented the best results. The same occurred in the bending load (in-plane shear - case 2) problems applied for straight beam with enrichment in domain/on boundary. Even the great disturbance visualized in 4-node MLGFM traditional technique is significantly better if compared to the plate elements tested by MacNeal and Harder (1985) but, on the other hand, reveals the low accuracy for shear problems using only the boundary enrichment method.

Table 2. Straight cantilever beam displacement results comparison

PROBLEM A (REGULAR ELEMENT)								
CASE TYPE	TIP LOAD	THEORETICAL*	CLASSIC FEM	MLGFM		ENRICHED MLGFM		
				4-NODE	9-NODE	DOMAIN/ BOUNDARY	DOMAIN	BOUNDARY
CASE 1	EXTENSION	3,0000E-05	2,9863E-05	2,9863E-05	2,9943E-05	2,9925E-05	3,0614E-05	2,9863E-05
CASE 2	IN-PLANE SHEAR	-1,0810E-01	-1,0088E-02	-1,0088E-02	-1,0703E-01	-1,0755E-01	-1,0755E-01	-1,0088E-02

PROBLEM B (TRAPEZOIDAL ELEMENT)								
CASE TYPE	TIP LOAD	THEORETICAL*	CLASSIC FEM	MLGFM		ENRICHED MLGFM		
				4-NODE	9-NODE	DOMAIN/ BOUNDARY	DOMAIN	BOUNDARY
CASE 1	EXTENSION	3,0000E-05	2,9927E-05	2,9927E-05	3,0338E-05	2,9917E-05	3,0638E-05	2,9839E-05
CASE 2	IN-PLANE SHEAR	-1,0810E-01	-2,6877E-03	-2,6860E-03	-7,1101E-02	-1,0747E-01	-1,0746E-01	-9,4011E-03

PROBLEM C (PARALLELOGRAM ELEMENT)								
CASE TYPE	TIP LOAD	THEORETICAL*	CLASSIC FEM	MLGFM		ENRICHED MLGFM		
				4-NODE	9-NODE	DOMAIN/ BOUNDARY	DOMAIN	BOUNDARY
CASE 1	EXTENSION	3,0000E-05	2,9919E-05	2,9918E-05	3,0313E-05	2,9925E-05	3,0614E-05	2,9863E-05
CASE 2	IN-PLANE SHEAR	-1,0810E-01	-3,4277E-03	-3,4274E-03	-4,9135E-02	-1,0755E-01	-1,0755E-01	-1,0088E-02

* MacNeal and Harder, 1985.

3.2 Thick-walled cylinder application

In this session, the thick-walled cylinder problem is used to test the EMLGFM. Note that, for this case, a plane strain state condition is assumed. The main goal here is to test the effect of nearly incompressible material. For this example, we have only used the enrichment approach domain/boundary at the same time. Four levels of enrichment are used here (from $p=2$ to $p=5$). Its geometry, mesh and properties are showed in “Fig. 6”.

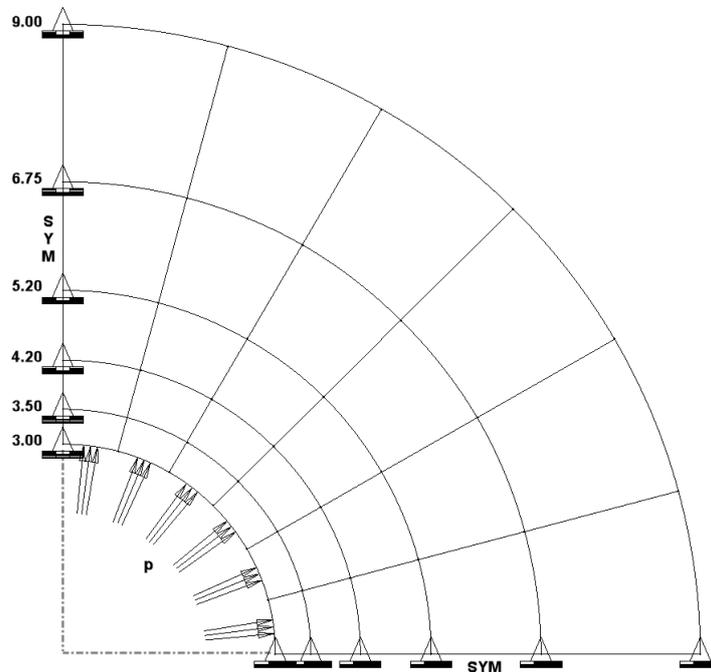


Figure 6. Thick-walled cylinder. Inner radius = 3.0; outer radius = 9.0; thickness = 1.0; $E = 1000$ problem $\nu = 0.49, 0.499, 0.4999$; plane strain condition; mesh = 5×6 . Loading: unit pressure at inner radius.

In Tab.4 (displacements results) and Tab. 5 (normal stress results), It's clearly verified the traditional 4-node element $p=1$ fails both in displacement and stress results. The nine-node element shows good results, but it is degrading as the poison decimal increases. The locking behavior was also observed by MacNeal and Harder (1985) when using traditional FEM. On the other hand, the EMLGFM presents excellent results even with different poison decimal numbers and reaching better results when increasing the polynomial degree as expected.

Table 4. Thick-walled cylinder displacement results comparison

Analytic Solution				
		$\nu = 0,49$	$\nu = 0,499$	$\nu = 0,4999$
displacement u at $r = 3.0$		0,0050399	0,0050602	0,0050623
30 ELEM				
Degree	Element	Displacement		
		$\nu = 0,49$	$\nu = 0,499$	$\nu = 0,4999$
$p=1$	FEM	0,00428952	0,00182865	0,00027089
	4-node	0,00423479	0,00180640	0,00026766
	9-node	0,00503330	0,00499420	0,00447490
$p=2$		0,00496845	0,00494931	0,00484208
$p=3$	4-node	0,00498654	0,00500047	0,00499993
$p=4$		0,00499756	0,00501670	0,00501851
$p=5$		0,00500314	0,00502228	0,00502416

Table 5. Thick-walled cylinder normal stress results comparison

σ_r (Normal Stress)								
Radius	Analytic	FEM	9-node	4-node				
				$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
9,00	0,25000	0,38745	0,41216	0,28239	0,21307	0,22288	0,23499	0,23788
6,75	0,34722	0,19593	0,54721	0,51501	0,34590	0,33924	0,33075	0,33023
5,20	0,49945	0,12744	0,73979	0,51548	0,48483	0,48360	0,48271	0,48038
4,20	0,69898	0,22443	0,97963	0,73724	0,68503	0,68269	0,68131	0,68198
3,50	0,95153	0,76670	1,26000	0,60080	0,91532	0,90483	0,92854	0,93280
3,00	1,25000	0,81921	1,59710	2,68125	1,31590	1,36185	1,32414	1,31451

4. CONCLUSION

This paper presented the first results for MLGFM using an enriched method to enrich the domain, the boundary or both at the same time for plane stress state. For plane strain state, the domain and boundary enrichment at the same time was used. The approach proposed here uses Lobatto shape functions as hierarchic shape functions since these functions naturally obey the delta property as in the context of FEM (only one shape function is non-null at each node).

The examples studied here indicate a strong potential of the Enriched MLGFM using both regular and distorted course mesh. Its high accuracy is first verified in extension straight beam as foreseen but more expressive for shear problems. Also, the thick-walled cylinder problem studied here showed good results demonstrating the great potential of EMLGFM in nearly incompressible materials.

Once quoted that, it's really important to point out the high possibilities of this method since only one level of enrichment was used on the examples explored in this paper. There is a huge field to be explored with MLGFM when it concerns to apply other enriched methods. For instance: other hierarchic level of enrichments, other shape functions, their numerical stability and convergence rate are consequently object for future researches as well as other problems in continuum mechanics.

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