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DYNAMICAL ANALYSIS AND CONTROL OF A PARAMETRICALLY EXCITED ELASTIC PENDULUM

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Abstract. *This work presents the modeling and a numerical method that enables the stability analysis and control of a periodic dynamic system, a parametrically excited elastic pendulum, with vertical excitation and variable length. First, the system is modeled through the Lagrangian method in order to obtain the movement equations. With this, it is designed a nonlinear SDRE (State-Dependent Riccati Equation) control to take the system to an equilibrium point and to an equilibrium periodic orbit. Then, a stability analysis of the nonlinear system is done through a stability diagram, which is possible to verify the occurrence of bifurcations for given parameter values. This method is based on L-F (Lyapunov-Floquet) transformation and uses the Chebyshev polynomial expansion to approximate the periodical term and to prevent ill-conditioning problem. The Picard iterative method is also used to calculate the state transition matrix (STM). If the system is proved to have bifurcations for a certain parameter values, a feedback and feedforward linear control based on Sinha's method is also designed to try to control the system. The stability analysis and control results are verified in numeric and symbolic simulations with MatlabTM.*

Keywords: *Periodic systems, L-F transformation, Chebyshev polynomials, Picard iteration, Control.*

1. INTRODUCTION

Systems described by nonlinear differential equations with time-periodic coefficients have great importance in widely diverse fields of engineering, physics, quantum mechanics and so on (Peruzzi *et al.*, 2006, 2016) It is well known that exact solutions of systems with periodic coefficients are limited; in general, only approximate solutions are possible to obtain.

A parametrically excited elastic pendulum is an example of these dynamic systems with time-periodic coefficients (Belyakov and Seyranian, 2012). In Fig.(1), l is the spring length at rest, z is the spring deformation, k is the spring linear coefficient, ϵ is the spring nonlinear coefficient, θ is the angular displacement, m the pendulum mass and u the external or vertical excitation.

The mathematical modeling was made through Lagrangian analysis, yielding the kinetic and potential energy equations of the system, to find the system movement equations (Andrade, 2003).

The Lagrangian of the system is described as

$$L = T - V = \frac{1}{2}m(\dot{z}^2 + \dot{u}^2 + (l+z)^2\dot{\theta}^2 + 2\dot{z}\dot{u}\cos\theta - 2\dot{u}(l+z)\dot{\theta}\sin\theta) + mg((l+z)\cos\theta - u) + (k\frac{z^2}{2} + \epsilon\frac{z^4}{4}) \quad (1)$$

In order to find the equations of motion that describe the system behavior, the Lagrangian equations are presented in terms of the angular displacement θ and the spring deformation z .

Considering the parametrically vertical excitation described as $u = A\sin(\omega t)$, with A being the excitation amplitude and ω the excitation frequency, we can write

$$\ddot{\theta} + \frac{2\dot{z}\dot{\theta}}{(l+z)} + \frac{A\omega^2\sin(\omega t)\sin\theta}{(l+z)} + \frac{g\sin\theta}{(l+z)} = 0 \quad (2)$$

$$\ddot{z} - (A\omega^2 \sin(\omega t) \cos\theta) - (l+z)\dot{\theta}^2 - g\cos\theta - (\omega_0^2 z + \frac{\epsilon z^3}{m}) = 0 \quad (3)$$

The Eq.(2) and Eq.(3) represent the equations of motion of the parametrically excited elastic pendulum.

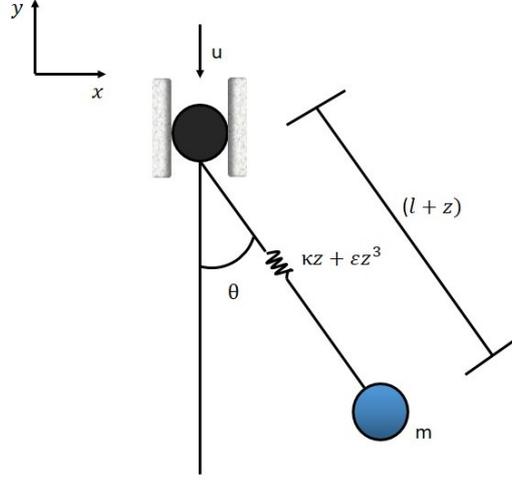


Figure 1: Elastic pendulum with vertical parametrically excitation.

2. SDRE CONTROL

Nowadays there exists a great variety of techniques for the synthesis of control laws for nonlinear systems. The SDRE (State-Dependent Riccati Equation) control theory is an efficient approach for the nonlinear feedback control that allows nonlinearities in the system states. This method can be considered as a nonlinear LQR control method, where the dynamic and weight matrices are functions of the states (György *et al.*, 2016).

One important advantage of using SDRE control technique is the opportunity to make different choices between control effort and state errors by choosing some defined penalty matrices, possibly as functions of the states (Erdem and Alleyne, 2004).

A general SDRE control problem consists of the infinite-horizon regulation. Given the equations of the system

$$\dot{x} = f(x) + g(x)u, \quad (4)$$

and a performance index which allows the trade off between state error x with the control input u via weighting matrices $\mathbf{Q}(\mathbf{x}), \mathbf{R}(\mathbf{x}) > 0 \quad \forall x$, as follows:

$$J = \frac{1}{2} \int_0^{\infty} (\mathbf{x}^T \mathbf{Q}(\mathbf{x}) \mathbf{x} + \mathbf{u}^T \mathbf{R}(\mathbf{x}) \mathbf{u}) dt, \quad (5)$$

The Eq.(4) can be rewritten as

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u}, \quad (6)$$

where $f(x) = \mathbf{A}(\mathbf{x})\mathbf{x}$ and $g(x) = \mathbf{B}(\mathbf{x})$. The choice of matrix $\mathbf{A}(\mathbf{x})$ is not unique, with different possible parametrizations. As the LQR formulation, the equation of the state feedback controller is

$$\mathbf{u}(\mathbf{x}) = -\mathbf{K}(\mathbf{x})\mathbf{x} = -\mathbf{R}(\mathbf{x})^{-1}\mathbf{B}(\mathbf{x})^T, \mathbf{P}(\mathbf{x})\mathbf{x} \quad (7)$$

where $\mathbf{P}(\mathbf{x})$ is a unique, positive-definite solution of the algebraic Riccati equation

$$\mathbf{P}(\mathbf{x})\mathbf{A}(\mathbf{x}) + \mathbf{A}(\mathbf{x})^T\mathbf{P}(\mathbf{x}) - \mathbf{P}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}(\mathbf{x})^{-1}\mathbf{B}(\mathbf{x})^T\mathbf{P}(\mathbf{x}) + \mathbf{Q}(\mathbf{x}) = 0 \quad (8)$$

At each instant or iteration, this method treats the state dependent coefficient matrices as being constant, computing the control by solving and LQ control problem (Çimen, 2008).

2.1 SDRE control application

For the dynamical nonlinear system treated in this work, first a parametrization is made before the design of the controller. This parametrization is useful in order to avoid using the real values of the system parameters, since every pendulum has its own physical values. To make the parametrization of Eq.(2) and (3), the following relations are introduced: $\bar{\tau} = \sqrt{\frac{g}{l}}t = \nu t$ and $Z = lz$. Thus, the equations of motion in dimensionless form are

$$\ddot{\theta} + \frac{2\dot{Z}}{1+Z}\dot{\theta} + (\alpha\Omega^2 \sin(\Omega\bar{\tau}) + 1) \frac{\sin\theta}{1+Z} = 0 \quad (9)$$

$$\ddot{Z} - (1+Z)\dot{\theta}^2 - (\alpha\Omega^2 \sin(\Omega\bar{\tau}) + 1)\cos\theta - \kappa Z - \lambda Z^3 = 0 \quad (10)$$

where $\alpha = \frac{A}{l}$, $\Omega = \omega\nu^{-1}$, $\lambda = \epsilon^2 l^2 \nu^{-2}$, $\kappa = \omega_0^2 \nu^{-2}$, with $\omega_0 = \sqrt{\frac{k}{m}}$.

With the parametrization done, now the Eq.(9) and (10) are rewritten in the space state form, in a system with four first order differential equation with the state variables $(x_1, x_2, x_3, x_4) = (\theta, \dot{\theta}, Z, \dot{Z})$. Now, the parametric movement equations of the system in state space for are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{(P\sin(\Omega\bar{\tau}) + 1)\sin x_1}{1+x_3} - \frac{2x_4 x_2}{1+x_3} \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= (P\sin(\Omega\bar{\tau}) + 1)\cos x_1 + (1+x_3)x_2^2 + \kappa x_3 + \lambda x_3^3 \end{aligned} \quad (11)$$

A control is designed introducing a torque signal $u = [u_1 \ u_2]$ in the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{(P\sin(\Omega\bar{\tau}) + 1)\sin x_1}{1+x_3} - \frac{2x_4 x_2}{1+x_3} + u_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= (P\sin(\Omega\bar{\tau}) + 1)\cos x_1 + (1+x_3)x_2^2 + \kappa x_3 + \lambda x_3^3 + u_2 \end{aligned} \quad (12)$$

The Eq.(12) is now written in the matrix form of the Eq.(6), with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(P\sin(\Omega\bar{\tau}) + 1)\sin(x_1)}{(1+x_3)x_1} & 0 & 0 & -\frac{2x_2}{1+x_3} \\ 0 & 0 & 0 & 1 \\ \frac{(P\sin(\Omega\bar{\tau}) + 1)\cos(x_1)}{x_1} & (1+x_3)x_2 & k + \lambda x_3^2 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (13)$$

and control matrix calculated using MatlabTM command $K = lqr(A, B, Q, R)$. The matrices Q and R defined as

$$\mathbf{Q} = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 0,1 & 0 \\ 0 & 0,1 \end{bmatrix} \quad (14)$$

To test the robustness of this kind of controller, all parameters $\bar{P} = \pm 0, 2P$, $\bar{\Omega} = \pm 0, 2\Omega$, $\bar{\kappa} = \pm 0, 2\kappa$ and $\bar{\lambda} = \pm 0, 2\lambda$ were submitted through 20% of uncertainty (a normally distribution) in their values. The value of $\pm 20\%$ is reasonable because the spring length and the pendulum frequency can be tested through a good variable range.

Two references was given for the controller. First, the system is taken to the origin (the equilibrium point $[0, 0, 0, 0]$), as seen in Fig.(2). Second, the reference is the periodic orbit $[\theta = 1; z = 2 + \cos(0, 5t)]$, as seen in Fig.(3).

3. STABILITY ANALYSIS

Given a periodic dynamical system, a structural stability analysis is usually carried out. This analysis is performed by finding the parameter values that make the system with periodic coefficients critical, in other words, find the parameter range in which a periodic solution is stable and unstable. The advantage of this approach is that the chaotic behavior can be found, describing the type of bifurcations that the system may have.

An efficient computational scheme for the analysis of periodic systems was first introduced in (Sinha *et al.*, 1979) and improved in (Sinha and Wu, 1991). The analysis is based on the idea that the fundamental solution matrix can be expanded in orthogonal polynomials, specifically shifted Chebyshev polynomials, over the principal period, solved by Picard iterations.

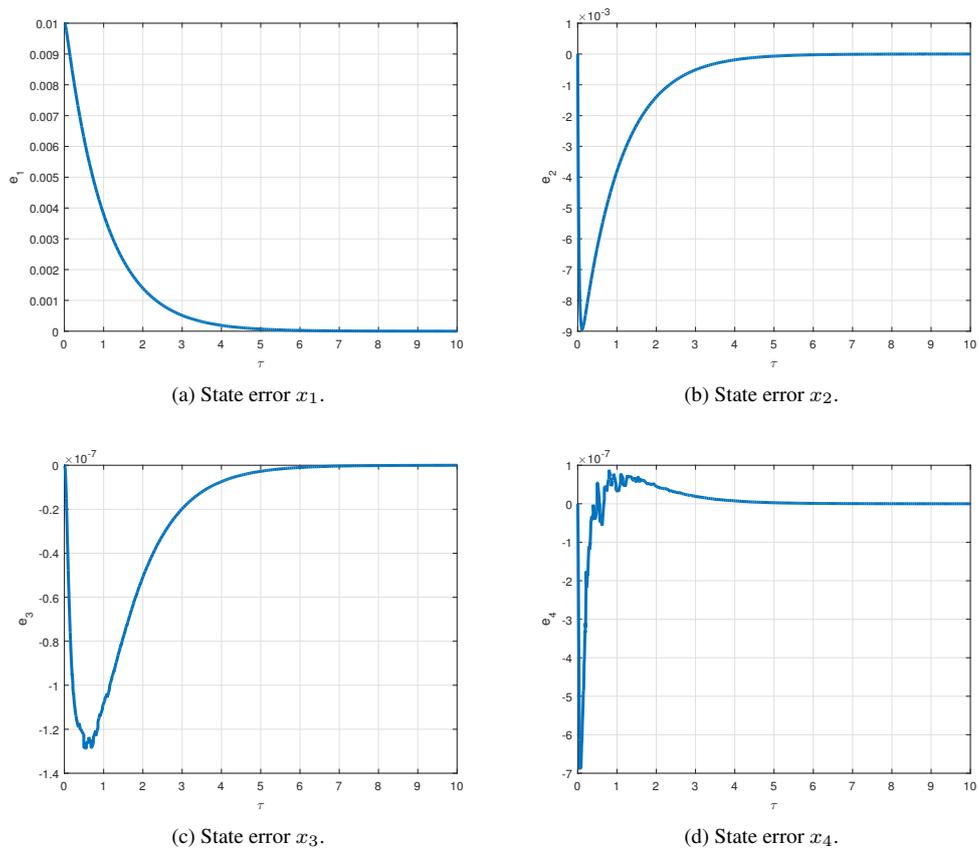


Figure 2: Feedback control error for the equilibrium point.

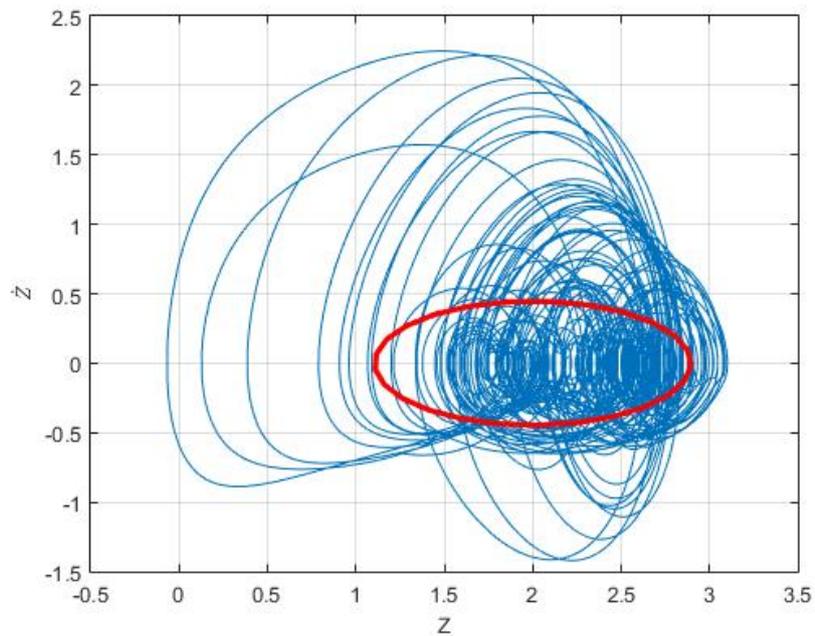


Figure 3: Phase portrait of the uncontrolled and controlled elastic pendulum for the periodic orbit.

3.1 Floquet theory

The Floquet theory gives conditions to guarantee the stability of a linear periodic system, where the stability analysis is made through the eigenvalues of the state transition matrix solved in the main period. The Floquet theory relies on the transformation of a linear time-periodic to a linear time-constant system.

Given a linear periodic system

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}, \quad (15)$$

where \mathbf{x} is an $n \times 1$ vector and $\mathbf{A}(t)$ is an $n \times n$ matrix of the principal period T , in which

$$\mathbf{A}(t + T) = \mathbf{A}(t) \quad (16)$$

Let $\phi(t)$ be a fundamental matrix solution of Eq.(15), i.e., a non-singular matrix where each of its columns is a solution, with $\phi(0) = \mathbf{I}$. Then, $\phi(t)$ satisfies Eq.(15), as follows

$$\dot{\phi}(t) = \mathbf{A}(t)\phi(t), \quad (17)$$

as well as

$$\phi(t + T) = \phi(t)\mathbf{F}, \quad (18)$$

where \mathbf{F} is a constant matrix.

A real number α_i is also defined as the real part of the characteristic exponent,

$$\alpha_i = \frac{1}{T} \ln |\mu_i|, \quad (19)$$

where μ_i are the eigenvalues of \mathbf{F} , called *characteristic exponent* or *Floquet multipliers*. From Eq.(18) the matrix \mathbf{F} is obtained directly

$$\mathbf{F} = \phi^{-1}(\mathbf{t})\phi(\mathbf{t} + \mathbf{T}) = \phi(\mathbf{0})\phi(\mathbf{T}) = \phi(\mathbf{T}) \quad (20)$$

The knowledge of $\phi(T)$ is sufficient to predict the stability of the system, being stable if $\alpha_i < 0$ or $\mu_i < 1$, for $i = 1, 2, \dots, n$. If $\mu_i = 1$, the system have periodic solutions.

3.2 Chebyshev polynomials

The expansion of the periodic matrices in terms of shifted Chebyshev polynomials reduces the periodic system to a set of algebraic linear equations, which solution in the interval of the main period can be obtained. The shifted Chebyshev polynomials of the first kind used are defined in terms of the standard Chebyshev polynomials, over the interval $[-1, 1]$, making the change variable

$$t \rightarrow \frac{t + 1}{2} \quad (21)$$

then, the shifted Chebyshev polynomials of the first kind are given by

$$T_r^*(t) = T_r(2t - 1), \quad 0 \leq t \leq 1 \quad (22)$$

valid over the interval $[0, 1]$. Thus, a continuous time function expanded into a Chebyshev series is calculated as

$$f(t) = \sum_{r=0}^{\infty} a_r S_r^*(t), \quad (23)$$

where $S_r^*(t)$ represent the shifted Chebyshev polynomials, and the coefficients a_r can be obtained as

$$a_r = \frac{1}{\delta} \int_0^1 \omega(\tau) f(\tau) S_r^*(t) d\tau, \quad r = 0, 1, 2, 3, \dots \quad (24)$$

where $\omega(\tau)$ is the appropriate weight function and

$$\delta = \begin{cases} \frac{\pi}{2}, & r \neq 0 \\ \pi, & r = 0 \end{cases} \quad (25)$$

3.3 Picard iteration

The need to solve Chebyshev coefficients of the solution are completely avoided using the method of Picard iterations (Sinha and Butcher, 1997). These iterations are sequences of approximations that allow this technique for systems with higher dimension.

Given a differential equation

$$\frac{dx(t)}{dt} = f(x(t), t), \quad x(t_0) = x^0, \quad (26)$$

it can be solved integrating both sides such as

$$x(t) = x^0 + \int_{t_0}^t f(x(\tau), \tau) d\tau, \quad (27)$$

where τ is a dummy variable. Assuming the initial value $x^0(t) = x^0$ and inserting on the right side of Eq.(27), an approximation of $x^1(t)$ is obtained. If this result is inserted back into the integral, it can generate the second approximation $x^2(t)$. This process is called Picard iteration, described as

$$x^{(k+1)}(t) = x^0 + \int_{t_0}^t f(x^k(\tau), \tau) d\tau, \quad (28)$$

resulting in a sequence of approximations to the real solution $x(t)$.

3.4 Stability diagram

To analyze the structural stability of a system, it is necessary to find the parameter values that make the system with periodic coefficient critical, i.e., a system with characteristics exponents equal 1 or -1 . The graphic obtained by the variation of the parameter values is called stability diagram. Through the analysis of this diagram it is possible to observe areas of stability and instability, as well as the type of bifurcations that may occur.

For the elastic pendulum, to plot the stability diagram, the Sinha's technique described in the previous sections is implemented. In the approximation to find the state transition matrix, the periodic matrix is expanded in 20 Chebyshev polynomials ($m = 20$) solved by 40 Picard iterations ($p = 40$). These numbers are chosen in order to satisfy the results precision in the approximation performed by the computational procedure, and are mainly used in literature (Peruzzi, 2005). The expression of the matrix calculated in the final of the main period is written as

$$\Phi^{p,m}(\tau, \alpha) = \hat{\mathbf{T}}^T(\tau) [\hat{\mathbf{I}} + (\sum_{k=1}^p \mathbf{L}(\alpha)^{k-1}) \mathbf{P}(\alpha)] = \hat{\mathbf{T}}^T(\tau) \mathbf{B}(\alpha), \quad (29)$$

where α represents the parameters of the system, τ is a dummy variable, $\hat{\mathbf{T}}(\tau)$ are the Chebyshev polynomials and $\mathbf{B}(\alpha)$ represent the Chebyshev coefficients of $\Phi(\tau, \alpha)$.

Making $(\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4) = (0, 0, 0, 0)$, the equilibrium point of Eq.(11) are determined as

$$P_1 = \begin{cases} x_1^* = 0 \\ x_2^* = 0 \\ x_3^* = 2\sqrt[3]{-\frac{1}{2\lambda}} \\ x_4^* = 0 \end{cases} \quad (30)$$

The Eq.(11) are linearized around this equilibrium point, yielding the Jacobian matrix

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{P \sin(\Omega \bar{\tau}) + 1}{1 + x_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \kappa + 3\lambda x_3^2 & 0 \end{bmatrix} \quad (31)$$

The linear system of Eq.(31) has period of $\frac{2\pi}{\Omega}$. With the normalization $\tau = \frac{2\pi}{\Omega} \bar{\tau}$, the linear system is rewritten as

$$\dot{\mathbf{X}} = \mathbf{J}\mathbf{X} \quad (32)$$

with a linear expansion of $\mathbf{J} = \mathbf{A}_1 f_1(\tau) + \mathbf{A}_2 f_2(\tau)$ in order to expand the functions in Chebyshev polynomials. Therefore, the periodic matrices are

$$\mathbf{A}_1 = \frac{2\pi}{\Omega} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{1+x_3} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & k + 3\lambda x_3^2 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \frac{2\pi}{\Omega} \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{P}{1+x_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (33)$$

with periodic functions

$$f_1(\tau) = 1, \quad f_2(\tau) = \sin(2\pi\tau) \quad (34)$$

The Fig.(4) shows the elastic pendulum stability diagram for the parameters P and Ω , where the black dots represent fold bifurcation and the red dots represent flip bifurcation. The values to the other parameters are $\kappa = 1, 5$ and $\lambda = -0.5$.

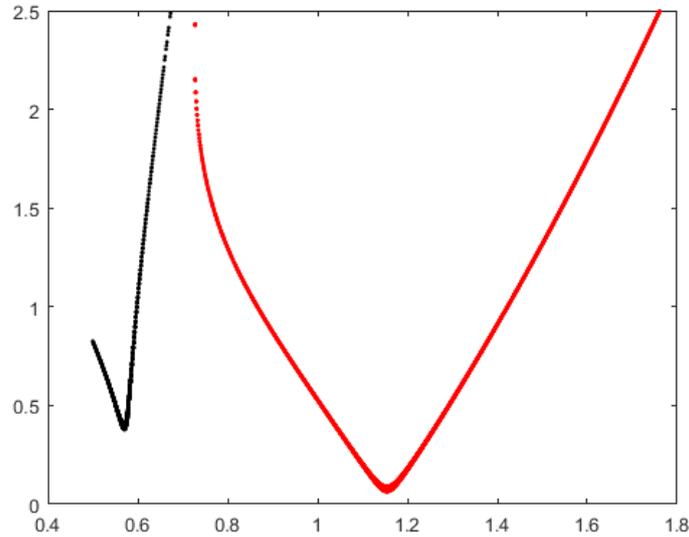


Figure 4: Elastic pendulum stability diagram, for $0 < P < 2, 5$ and $0, 5 < \Omega < 2$.

For values of $P = 1$ and $\Omega = 1, 22$, the approximation of matrix $\phi(1)$ is

$$\phi^{40,20}(1) = \begin{bmatrix} -1,909 & -0,042 & 0 & 0 \\ -0,346 & -0,643 & 0 & 0 \\ 0 & 0 & -0,051 & -0,063 \\ 0 & 0 & 0,285 & 0 \end{bmatrix} \quad (35)$$

having the following eigenvalues (characteristics exponents)

$$\begin{aligned} \mu_1 &= -1,920 \\ \mu_2 &= -0,631 \\ \mu_3 &= -0,025 + 0,132i \\ \mu_4 &= -0,025 - 0,132i \end{aligned} \quad (36)$$

The presence of $|\mu_1| = 1,920 > 1$ indicates that the system is unstable for these parameter values.

4. SINHA'S CONTROL METHOD

It was proved in the latter section that for $P = 1$ and $\Omega = 1, 22$, the elastic pendulum presents flip bifurcation. To control this chaotic behavior, it is necessary to design a control law combining a feedforward and a feedback controller $\mathbf{u}(t) = \mathbf{u}_f(t) + \mathbf{u}_t(t)$, to conduct the unstable orbit (for a given parameter values) to a periodic desired orbit and/or desired point (Sinha *et al.*, 2005). First, the periodic desired orbits chosen was $y_1(t) = 5 + \cos(\omega t)$, $y_2(t) = \sin(\omega t)$, $y_3 = \sin(\omega t)$ and $y_4 = \sin(\omega t)$.

With the feedforward control \mathbf{u}_f and feedback control \mathbf{u}_t , Eq.(11) becomes

$$\begin{aligned} \dot{x}_1 &= x_2 + u_{f,1} + u_t \\ \dot{x}_2 &= -\frac{(P\sin(\Omega\bar{\tau}) + 1)\sin x_1}{1 + x_3} - \frac{2x_4x_2}{1 + x_3} + u_{f,2} \\ \dot{x}_3 &= x_4 + u_{f,3} \\ \dot{x}_4 &= (P\sin(\Omega\bar{\tau}) + 1)\cos x_1 + (1 + x_3)x_2^2 + \kappa x_3 + \lambda x_3^3 + u_{f,4} \end{aligned} \quad (37)$$

The feedforward control equations \mathbf{u}_f are given by

$$\begin{aligned} u_{f,1} &= \dot{y}_1 - f_1(y_1, t) \\ u_{f,2} &= \dot{y}_2 - f_2(y_2, t) \\ u_{f,3} &= \dot{y}_3 - f_3(y_3, t) \\ u_{f,4} &= \dot{y}_4 - f_4(y_4, t) \end{aligned} \quad (38)$$

and the feedback control \mathbf{u}_t equation is given by

$$\mathbf{u}_t = -\mathbf{k} [e_1 \ e_2 \ e_3 \ e_4]^T, \quad (39)$$

with $\mathbf{k} = [k_1 \ k_2 \ k_3 \ k_4]$ (Sinha and Dávid, 2006). The control input was assumed as $\mathbf{B} = [1 \ 0 \ 0 \ 0]^T$. Applying the feedback and feedforward controllers described in Eqs.(38) and (39), and linearizing Eq.(37) for the error variables, we obtain

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \Gamma & \Delta & \Theta & \Lambda \\ 0 & 0 & 0 & 1 \\ \Xi & \Pi & \Sigma & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_t, \quad (40)$$

with

$$\begin{aligned} \Gamma &= -\frac{(P\sin(\Omega\tau) + 1)\cos(5 + \cos(\omega t))}{1 + \sin(\omega t)} \\ \Delta &= -\frac{2\sin(\omega t)}{1 + \sin(\omega t)} \\ \Theta &= \frac{(P\sin(\Omega\tau) + 1)\sin(5 + \cos(\omega t))}{(1 + \sin(\omega t))^2} + \frac{1 + \cos(2\omega t)}{(1 + \sin(\omega t))^2} \\ \Lambda &= -\frac{2\sin(\omega t)}{1 + \sin(\omega t)} \\ \Xi &= -(P\sin(\Omega\tau) + 1)\sin(5 + \cos(\omega t)) \\ \Pi &= 2\sin(\omega t) + 2\sin(\omega t)^2 \\ \Sigma &= \sin(\omega t)^2 + \kappa + 3\lambda\sin(\omega t)^2 \end{aligned} \quad (41)$$

With the closed loop of Eq.(40), the matrix error is now a function of gains k_1 , k_2 , k_3 and k_4 . To apply the symbolic technique described in section 3.4, this matrix has to be expanded in linear terms of $\mathbf{A}_1 f_1 + \mathbf{A}_2 f_2 + \dots$. Therefore, the functions f obtained are very complex, not simply sines or cosines, and due to this they cannot be expanded in Chebyshev polynomials.

Since the control method was not able to be implemented for conduct the system to a desired periodic orbit, a desired point is now chosen as a simpler approach. The desired point selected was $y_1 = \frac{\pi}{6} \approx 0,5236$, $y_2 = 1$, $y_3 = 0,05$ and $y_4 = 0,05$. Following the same steps before, linearizing Eq.(37) for the error variables, we obtain now

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \Upsilon & \frac{8}{3} & \Phi & -\frac{2}{3} \\ 0 & 0 & 0 & 1 \\ \Psi & 6 & 1 + \kappa + 12\lambda & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_t, \quad (42)$$

with

$$\begin{aligned}\Upsilon &= -0,288868(P\sin(\Omega\tau) + 1) \\ \Phi &= 0,05556(P\sin(\Omega\tau) + 1) + 0,88889 \\ \Psi &= -0,5(P\sin(\Omega\tau) + 1)\end{aligned}\quad (43)$$

Now, the linear expansion of the closed loop error matrix in terms of $A_1 f_1 + A_2 f_2$ is

$$\mathbf{A}_1 = \frac{2\pi}{\Omega} \begin{bmatrix} -k_1 & 1 - k_2 & -k_3 & -k_4 \\ 0,28868 & \frac{8}{3} & \frac{8}{9} & -\frac{2}{3} \\ 0 & 0 & 0 & 1 \\ -0,5 & 6 & 1 + \kappa + 12\lambda & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \frac{2\pi}{\Omega} \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0,28868P & 0 & 0,05556P & 0 \\ 0 & 0 & 0 & 0 \\ -0,5P & 0 & 0 & 0 \\ , & & & \end{bmatrix}\quad (44)$$

with periodic functions

$$f_1(\tau) = 1, \quad f_2(\tau) = \sin(2\pi\tau)\quad (45)$$

The symbolic computation technique was applied with 20 Chebyshev polynomials and 20 Picard iterations to calculate the gains in order to control the system. It turns out that the computation did not give any results for the gain matrix k . One reasonable conclusion that can be inferred is the occurrence of bifurcation with the given parameters, i.e., there is no value of gain for the linear controller to control the system with bifurcation. This problem mainly occurs due to the extensible characteristic of the pendulum cable.

5. CONCLUSIONS

A modeling of a parametrically excited elastic pendulum with stability analysis and control has been proposed. Since the elastic pendulum is a nonlinear, non-autonomous, periodic system, a stability analysis was performed and two control techniques were implemented.

First, a simple nonlinear control technique was designed to take the system to equilibrium. An uncertainty of 20% was inserted to the system parameters, and the SDRE control proved to be a useful control technique.

Second, a stability analysis was made following the Sinha's method, in order to find bifurcations with variations of the parameters values. This analysis relies on approximating the fundamental solution matrix of the linearized system as a function of its parameters, using orthogonal polynomial expansion (Sinha *et al.*, 1979) of the periodic terms and Picard iterations to reduce the computation effort. A similar elastic pendulum modeling with stability analysis can be found in (Mesquita, 2007).

Last, a symbolic computational procedure to design a control to the linearized system with time periodic coefficients was made. This Sinha's method is also used to control nonlinear, non-autonomous systems to a desired periodic orbit or a desired point, which does not need to be a system's solution. The control vector is composed of two parts: a feedback equation and a feedforward equation. The linearization of the equations was made around a desired periodic orbit and a desired point. For the first case, the set of equations resulted in complex trigonometric functions, which were not possible to expand in terms of Chebyshev polynomials. In the second case, the symbolic calculation to find the Floquet transition matrix for the closed-loop system, it was not possible to find values for the gains. A reason for that is because for the given parameters values, the system presents flip bifurcation, a chaotic phenomena (Abed *et al.*, 1994). Since the Sinha' control method derives a linear control, it was not possible for the controller to find a gain that stabilizes the system's chaotic behavior.

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7. REFERENCES

- Abed, E.H., Wang, H.O. and Chen, R.C., 1994. "Stabilization of period doubling bifurcations and implications for control of chaos". *Physica D*, Vol. 70, pp. 154–164.
- Andrade, V.S., 2003. *Análise da Dinâmica Caótica de Pêndulos com Excitação Paramétrica no Suporte*. Ph.D. thesis, Universidade de São Paulo.

- Belyakov, A. and Seyranian, A., 2012. “Dynamics of a Pendulum of Variable Length and Similar Problems”. *Nonlinear, Bifurcation and Chaos*.
- Çimen, T., 2008. “State-Dependent Riccati Equation (SDRE) Control : A Survey”. In *IFAC Proceedings Volumes*. IFAC, Vol. 41, pp. 3761–3775. doi:10.3182/20080706-5-KR-1001.00635. URL <http://dx.doi.org/10.3182/20080706-5-KR-1001.00635>.
- Erdem, E. and Alleyne, A., 2004. “Design of a class of nonlinear controllers via state dependent Riccati equations”. *IEEE Transactions on Control Systems Technology*, Vol. 12, No. 1, pp. 133–137. doi:10.1109/TCST.2003.819588.
- György, K., Dávid, L. and Kelemen, A., 2016. “Theoretical Study of the Nonlinear Control Algorithms with Continuous and Discrete-Time State Dependent Riccati Equation”. *Procedia Technology*, Vol. 22, No. October 2015, pp. 582–591. doi:10.1016/j.protcy.2016.01.123.
- Mesquita, A.J.N., 2007. *Universidade estadual paulista*. Ph.D. thesis, Universidade Estadual Paulista - UNESP.
- Peruzzi, N.J., 2005. *Dinâmica Não Linear e Controle de Sistemas Ideais e Não Ideais Periódicos*. Ph.D. thesis, Universidade Estadual de Campinas.
- Peruzzi, N.J., Balthazar, J.M., Pontes, B.R. and Brasil, R.M.L.R.F., 2006. “Nonlinear Dynamics and Control of an Ideal/Nonideal Load Transportation System With Periodic Coefficients”. *Journal of Computation and Nonlinear Dynamics*, Vol. 2, No. January 2007, pp. 32–39. doi:10.1115/1.2389040.
- Peruzzi, N.J., Chavarette, F.R., Balthazar, J.M., Tuset, A.M., Peticarrari, A. and Brasil, R., 2016. “The Dynamic Behavior of a Parametrically Excited Time-Periodic MEMS Taking Into Account Parametric Errors”. *Journal of Vibration and Control*, , No. September 2014. doi:10.1177/1077546315573913.
- Sinha, S.C., Chou, C.C. and Denman, H.H., 1979. “Stability analysis of systems with periodic coefficients: An approximate approach”. *Journal of Sound and Vibration*, Vol. 64, No. 4, pp. 515–527. doi:10.1016/0022-460X(79)90801-0.
- Sinha, S.C., Gourdon, E. and Zhang, Y., 2005. “Control of time-periodic systems via symbolic computation with application to chaos control”. *Communications in Nonlinear Science and Numerical Simulation*, Vol. 10, pp. 835–854. doi:10.1016/j.cnsns.2004.06.001.
- Sinha, S.C. and Wu, D.H., 1991. “An Efficient Computational Scheme for the Analysis of Periodic Systems”. *Journal of Sound and Vibration*, Vol. 151, pp. 91–117.
- Sinha, S. and Butcher, E.A., 1997. “Symbolic Computation of Fundamental Time-Periodic Dynamical Systems”. *Journal of Sound and Vibration*, Vol. 206, pp. 61–85.
- Sinha, S. and Dávid, A., 2006. “Control of chaos in nonlinear systems with time-periodic coefficients”. *Philosophical Transactions of the Royal Society*, , No. July, pp. 2417–2432. doi:10.1098/rsta.2006.1832.

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