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NUMERICAL ANALYSIS OF ENTHALPY OF A COMPRESSIBLE FLOW OVER A FLAT PLATE

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Abstract. *This paper develops an outcome to momentum and energy equations of boundary layer providing the velocity and temperature stability and profile inside the boundary layer region, respectively. The results are obtained by variations in inlet freestream velocity over an adiabatic flat plate. The freestream is composed by air at sea-level and the variations in density and viscosity are negligible. Variations in the freestream Mach number and angle of attack of plate are made, from 0.3 to 0.9 and approximately -18° to 10°, respectively. Due to changes in flat plate angulation, the flow is considered to be Falkner-Skan type. The calculations revealed a good agreement with related literature. The proposed method uses a Python library to calculate the momentum and energy boundary layers using finite difference method. It displayed faster convergence even with greater grid refinement. There are no results for great variation in angle of attack due to convergence issues.*

Keywords: *boundary layer, Mach number, Falkner-Skan, Pyhton*

1. INTRODUCTION

Computational methods are vastly used as tool in Fluid Mechanics. These methods use mathematical modeling and numeric methods to achieve results, for Fluid Mechanics these techniques are known as CFD (Computational Fluid Dynamics). It provided great enhancement to the field, particularly in Aerodynamics because of the phenomena involving turbulence (Anderson, 2001). One approach is the iteration of a wall from a body submerse in a fluid with the fluid. A region that the viscosity of fluid determinates the flow behavior formed between the wall and the freestream fluid, known as Boundary layer. It has a great influence in drag force due to viscous effects, thus it became a major factor in Aerodynamic efficiency.

This paper provides an approach of analysis of a compressible flow below supersonic regime over a flat plate. It is used finite difference method to evaluate the velocity and temperature profiles inside the boundary layer region. The present work utilized an alternative method to computing the momentum and energy equations of the boundary layer. It used the libraries of ODE (Ordinary Differential Equation) solving in Python to compute the outcome of the equations of boundary layer, thus displaying their velocity and temperature profiles. Python is a high-end language that appeared in 1990 with propose to help the programmer because is easier than the others languages at that time. At present time, it is use for science and engineering computation because is simple and has great libraries of mathematical calculation (Kiusalaas, 2013).

1.1 Mathematical Modeling

There are three equations that describe the physical behavior of a fluid, the continuity, momentum and conservation of energy and they are showed below.

$$\partial\rho/\partial t + div\ \rho V = 0 \tag{1}$$

$$\rho\, DV/dt = \rho g - \nabla P + \partial/\partial x_j\, [\mu(\partial u_i/\partial x_j + \partial u_j/\partial x_i) + \delta_{ij}\lambda\ div\ V] \tag{2}$$

$$\rho(Dh/Dt) = DP/Dt + div(k\nabla T) + \Phi \tag{3}$$

Before performing the transformations that result in the boundary layer equations, simplifications are made in the government equations. Some hypotheses are steady state flow, 2D, specific heat at constant volume and constant pressure are constants, the boundary region is thin making the pressure variation in the perpendicular direction of the flat plate is negligible and $v << u$, $\partial/\partial x << \partial/\partial y$ (White, 1991). These hypotheses lead to a new set of equations:

$$\partial(\rho u)/\partial x + \partial(\rho v)/\partial y = 0 \tag{4}$$

$$\rho u(\partial u/\partial x)+\rho v(\partial u/\partial y)=- dp_{\infty}/dx + \partial/\partial y\,[\mu(\partial u/\partial y)] \tag{5}$$

$$\rho u(\partial h/\partial x)+\rho v(\partial h/\partial y)=u\left(dp_{\infty}/dx\right)+(\partial/\partial y)[(\mu/Pr)(\partial h/\partial y)]+\mu(\partial u/\partial y)^2 \tag{6}$$

These equations are presented from the relations of Crocco (1932) and Busemann (1931). This works followed the Illingworth transformation (1950) which writes the density times velocity as derivation of stream function (White, 1991). However, this definition violates the mass conservation because it has a negative term multiplying the density. To avoid this problem, Illingworth substituted y variable for $\int \rho dy$. Now, the stream function is depended of three variables: x, y, ρ. Using Mathematic tool similarity it is wrote the three variables into two new variables ξ and η. The viscous effect should be associated with ξ and the density effects associated with η (White, 1991), thus one can write the stream function as

$$\Psi(\xi,\eta)=\int \rho u\, dy=G(\xi)f'(\eta) \tag{7}$$

The variables ξ and η can be wrote as

$$\xi = \int_0^X \rho_{\infty}(x)U_{\infty}(x)\mu_{\infty}(x)\, dx \tag{8}$$

$$\eta = \frac{U_{\infty}}{\sqrt{2\xi}}\int_0^Y \rho\, dy \tag{9}$$

Using Eq. (7), Eq. (8) and Eq. (9) into Eq. (5), one can found the momentum equation of boundary layer. So the final form of this equation is

$$\left(C_R f''\right)' + ff'' + (2\xi/U_{\infty})(dU_{\infty}/d\xi)\left[(\rho_{\infty}/\rho) - f'^2\right] = 0 \tag{9}$$

where C_R is $\rho\mu/\rho_{\infty}\mu_{\infty}$ and f' is u/U_{∞}.

Since the flows evaluated in this paper are Falkner-Skan type a modification is needed in the Eq. (9). The term involving derivation of freestream velocity by variable ξ is substituted by a term that contains the variation in angulation of flat plate, thus:

$$U_{\infty}=kx^m \tag{10}$$

where k is a flow geometry coefficient and m is an angular coefficient of flat plate, this equation describes Falkner-Skan flows.

Substituting the terms of Eq. (9) with Eq. (10) is obtained the follow equation:

$$\left(C_R f''\right)' + ff'' + [2m/(m+1)]\left[(\rho_{\infty}/\rho) - f'^2\right] = 0 \tag{11}$$

where m is an angulation coefficient, it determinates the orientation of flat plate, when negative the plate is rotating in clockwise direction, whereas when positive is in anti-clockwise.

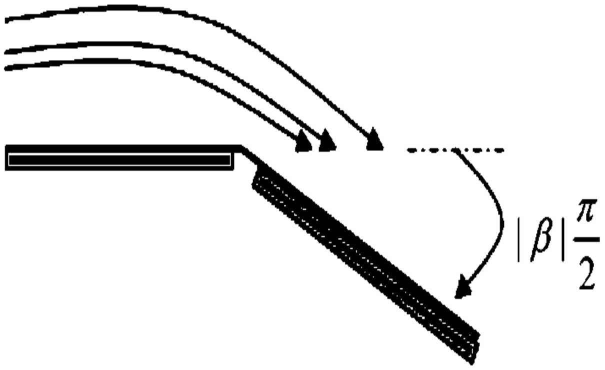


Figure 1. Flow over a wedge. Source: BARARNIA *et al*. (2012).

For the energy equation of boundary layer is used the same procedure. However the equation that substitutes it terms by the new variables is Eq. (6). Thus, the new equation is:

$$[(C_R/Pr)g']'+fg'=[(\xi/H_\infty)(dH_\infty/d\xi)]f'\left[2g+\left(U_\infty^2/h_\infty\right)f'^2\right]-(U_\infty^2/h_\infty)C_R f''^2 \quad (12)$$

where H_∞ is the total freestream stagnation enthalpy, $h_\infty + U_\infty^{\ 2}/2$, and the function g is the relation of local enthalpy and freestream enthalpy, $g = h(x,y) / h_\infty$.

The derivative term of H_∞ by ξ is zero because the variable is non-related of enthalpy. The term U_∞^2/h_∞ is a variable from Eckert number and can be wrote as Mach number variable (White, 1991), thus:

$$(C_R/Pr)'\ g''+fg'=-(\gamma-1)Ma_\infty^2 C_R f''^2 \quad (13)$$

where γ is c_P/c_V.

Since air is the fluid evaluated, changes can be made in Eq. (10) and Eq. (12) to simplify the equations. First, is assumed that the variation of local density and viscosity in relation to freestream density and viscosity is negligible, thus $C_R = 1$ and $Pr = 1$. Other simplification is approximate the relation ρ_∞/ρ to function g (White, 1991), thus:

$$f'''+ff''+[2m/(m+1)]\left[g-f'^2\right]=0 \quad (14)$$

where the term $[2m/(m+1)]$ is called β.

$$g''+fg'=-(\gamma-1)Ma_\infty^2 f''^2 \quad (15)$$

These are the final form of the equations of momentum and energy of boundary layer. The modifications were made firstly by Cohen e Reshotko (1956), which is specified for Falkner-Skan flows (White, 1991).

1.2 Numerical modeling

It cannot be solved numerically problems of superior orders of ODE's directly so is needed a change of all superior ODE's elements to became first order ODE's arguments and allocate them in an array (Pine, 2013). Solving the momentum equation and choosing the *f* function to derivate one can have:

$$f=z_0 \quad (16)$$

$$dz_0/d\eta=z_1 \quad (17)$$

where *z* is an arbitrary value of *f* derivate.

Following the same steps in derivation,

$$dz_1/d\eta=z_2 \quad (18)$$

However, Eq. (14) presents the third derivate of function *f* as:

$$f''' = -ff'' - \beta\left(g - f'^2\right) \tag{19}$$

$$dz_2/d\eta = z_3 = f''' \tag{20}$$

thus,

$$dz_2/d\eta = -z_0 z_1 - \beta(g - z_1^2) \tag{21}$$

Since both equations needed to be solved simultaneously:

$$g = z_3 \tag{22}$$

therefore,

$$dz_3/d\eta = z_4 = g'' = -fz_3 - Bf''^2 \tag{23}$$

where $B = -(\gamma - 1)\, Ma_\infty^2$.

By all the elements defined is allocate into a single array.

$$array = \left(\left[z_1,\ z_2,\ -z_0 z_1 - \beta\left(z_3 - z_1^2\right),\ z_4,\ -z_0 z_3 - B z_2^2\right]\right) \tag{24}$$

2. RESULTS AND DISCUSSION

Before the run of the calculations some parameters are established: $g(0) = 1$; c_P=cte; c_V=cte; and viscous effects are negligible.

First is showed the results of non-dimensional values that include the fluid acceleration.

Table 1. $f''(0)$ for several values of Mach number and m.

Mach Number	*m*								
	-0.0904	**-0.0800**	**-0.0500**	**-0.0300**	**0.0000**	**0.0300**	**0.0500**	**0.0800**	**1.0000**
0.3	0.0997*	0.1448	0.3077	0.3813	0.4696	0.5417	0.5835	0.6391	1.2543*
0.4	0.0954*	0.1408	0.3062	0.3804	0.4696	0.5424	0.5847	0.641	1.2646*
0.5	0.0899*	0.1357	0.3042	0.3793	0.4696	0.5434	0.5862	0.6434	1.2779*
0.6	0.0829*	0.1292	0.3017	0.378	0.4696	0.5445	0.5881	0.6463	1.2946*
0.7	0.0746*	0.1213	0.2988	0.3764	0.4696	0.5459	0.5904	0.6498	1.3148*
0.8	0.0647*	0.1118	0.2954	0.3746	0.4696	0.5475	0.593	0.6539	1.3387*
0.9	0.0529*	0.1004	0.2916	0.3725	0.4696	0.5493	0.5959	0.6584	1.3667*

Note: (*) data contain uncertainties due to problems of algorithm convergence.

The results presented great similarity with results found in literature, White (1991), Bararnia *et al*. (2012) and Mutlag *et* al. (2013) showed the same result for $m = 0$. The minimum value $m = -0.0904$ is select based in the boundary layer separation, whereas for positives values is infinity (Bejan and Kraus, 2003). Table 1 presented the acceleration function f'' is related to flow geometry than inlet velocity and it determinates the velocity profile and stabilization. Below it is plot function f'.

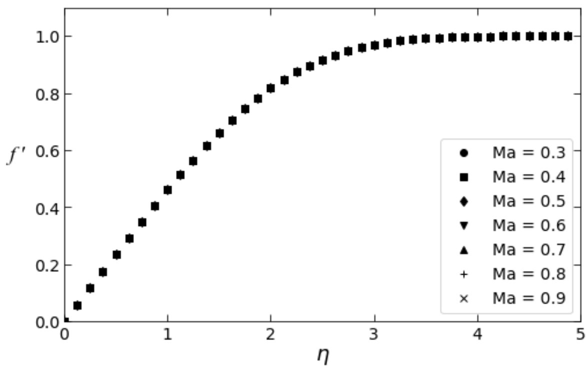


Figure 2. Plot of f' *versus* η for m = 0.

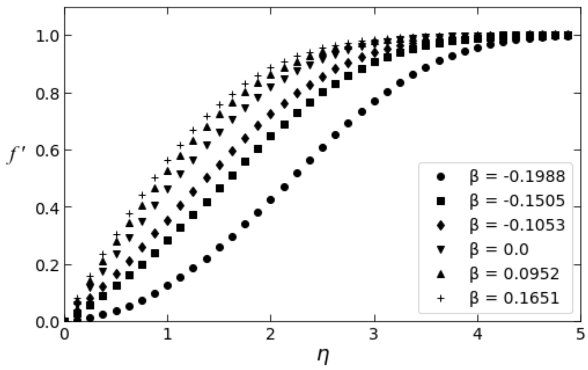


Figure 3. Plots of f' versus η for various β with $Ma = 0.3$

The curves have great compatibility with literature, Schlichting (1979) showed similar curves for function f'. Even though, η is a variable of two variables, x and y, it is the same direction of y.

2.1 Enthalpy Analysis

On the other hand, the temperatures profiles of boundary layer are strong related with the Mach number as one can see in the Eq. (15).

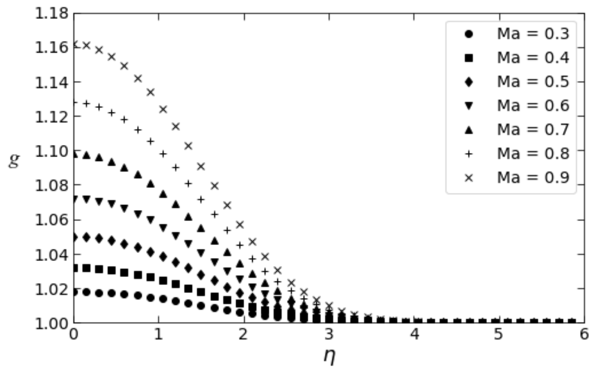
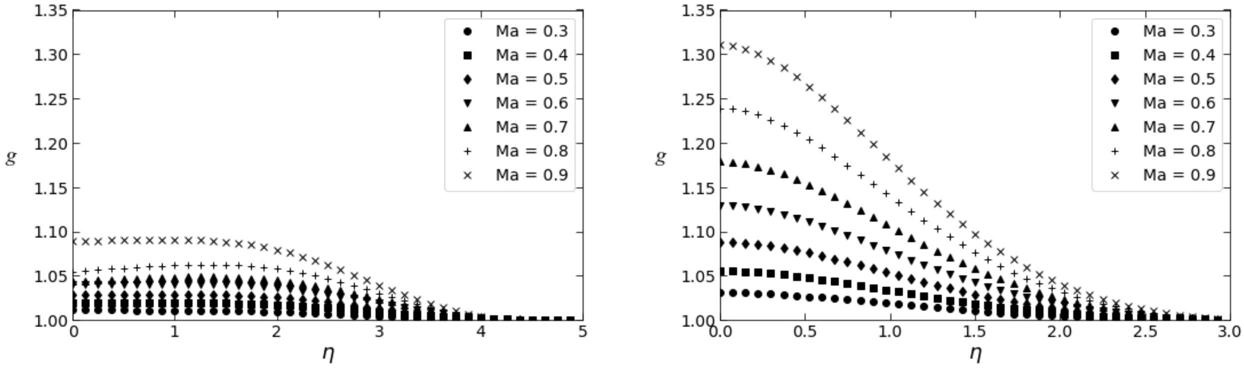


Figure 4. Plot of g *versus* η with $m = 0$.

Figure 4 shows most of the kinetic energy from the flow is transforming in thermal energy therefore heating the boundary layer region. Soo (1959) presented similar curves of temperature for flat plate as well similar curves for heat transfer, which is going to be comment below.



(a) (b)

Figure 5. Plots of g *versus* η with (a) m = -0.0904 and (b) m = 0.5

Thus, the temperature profile has influence of geometry too because Eq. (15) is depended of function f'' and as presented before it has great influence from geometry. Figure 5(a) illustrates for negative m the difference of enthalpy, hence temperature is small this indicates that the kinetic energy from the flow is transferring to another physical property or not, on the other hand Fig. 5(b) shows a larger energy transfer from kinetic to thermal form.

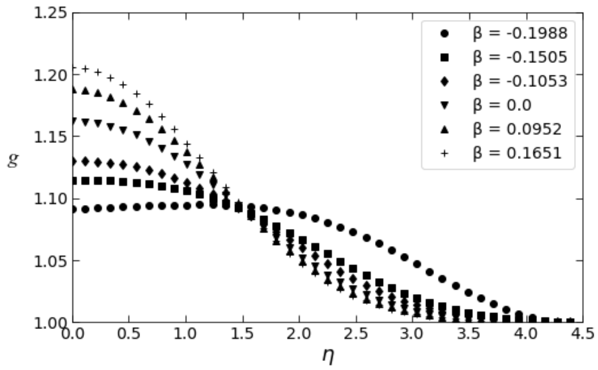


Figure 6. Plot of g *versus* η for various β with Ma = 0.9.

One can evaluate heat transfer occurs inside boundary layer in the viscous region that the profile velocity is closer to the freestream velocity. Figure 6 shows the geometry has influence in heat transfer location, moved to the positive direction of η.

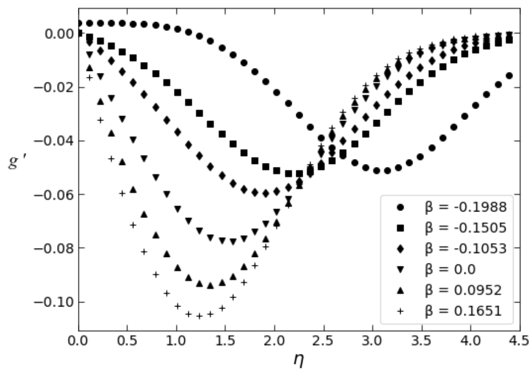
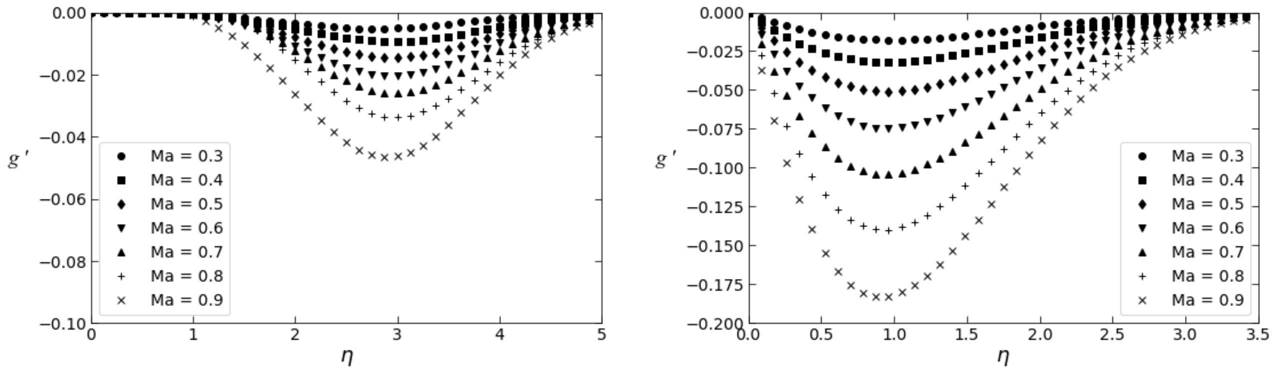


Figure 7. Plot of g' *versus* η for Ma = 0.9.

This indicates the shock wave formation occurs above the flat plate, inside the viscous region because is the region with bigger rate of heat transfer. It can be evaluated that geometry with positive values for m it is more likely to have shock wave than the geometries with negative values for m. Due to convergence issues η has a shorter range. Following the heat transfer evaluation, the function g' of the Fig. 5(a) and Fig. 5(b) is:



(a) (b)

Figure 8. Plots of g' *versus* η with (a) m = -0.0904 and (b) m = 0.5

Table 2. *g(0)* for several values of Mach number and *m*.

Mach Number	*m*								
	-0.0904	**-0.0800**	**-0.0500**	**-0.0300**	**0.0000**	**0.1000**	**0.3000**	**0.8000**	**1.0000**
0.3	1.0114	1.0123	1.0148	1.0162	1.0180	1.0225	1.0280	1.0345	****
0.4	1.0201	1.0218	1.0263	1.0288	1.0320	1.0401	1.0501	1.0621	1.0650*
0.5	1.0291	1.0338	1.0409	1.0448	1.0500	1.0630	1.0791	1.0986	1.1034*
0.6	1.0399*	1.0483	1.0586	1.0644	1.0720	1.0911	1.1152*	1.1448*	1.1522*
0.7	1.0439*	1.0745	1.0794	1.0875	1.0980	1.1248	1.1590*	1.2018*	1.2127*
0.8	1.0550*	1.0937	1.1032	1.1139	1.1280	1.1642	1.2111*	1.2712*	1.2866*
0.9	1.0891*	1.1053	1.1298	1.1436	1.1620	1.2095	1.2723*	1.3549*	1.3764*

Note: (*) data contain uncertainties due to problems of algorithm convergence.

The results showed the rate of heat transfer depends of the kinetic energy of fluid but it is location in the viscous region is determinate by the flow geometry. Since the geometry determinate the value of function f'', which is the acceleration of particles in the viscous region, thus influencing the location of high rate of heat transfer. Therefore, shock waves are likely to form in this region of the boundary layer because it has more energy than others areas, which is specific for this case of adiabatic plate.

3. CONCLUSION

We conclude this paper presented good results of momentum and energy equations of boundary layer, which has great similarity with literature. In literature most of the papers and books used innumerous finite difference methods with different convergence methods. This paper used a different approach, using Python library called *Scipy*. It is based in FORTRAN library of solving ODE's and convergence methods of Krylov approximation for inverse Jacobian and Anderson mixing. The outcomes have great stability and were fast in displaying the plots and values. However, some issues with convergence of Jacobian matrix were found with greater values of β, negatives or positives. Despite the issues with convergence in some cases, Python showed as excellent programming language for numeric analysis providing fast calculations.

SYMBOLS

β	Angulation coeffiecient
c_P	Specific heat at constant pressure
c_V	Specific heat at constant volume
C_R	Chapman-Rubesin parameter
δ	Kronecker Delta
g	Gravity acceleration
γ	Adiabatic expansion coefficient
h	Specific Enthalpy
k	Thermal conductivity
λ	Viscous Volumetric coefficient
m	Angulation coefficient
μ	Absolut viscosity
P	Pressure
Φ	Viscous dissipation rate
Pr	Prandtl number
ρ	Density
t	Time
T	Temperature
V	Total Velocity
x	Direction parallel to flat plate
y	Direction perpendicular to flat plate

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