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GENERALIZED *Q*-DISTRIBUTIONS IN RELIABILITY MODELING

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Abstract. *A comparison of four models of life distributions is applied in a practical reliability modeling. Two are q-type distributions of their usual ones, inspired by mathematical developments that followed nonextensive statistical mechanics, namely q-Weibull and q-exponential. Bernard's approximation of median ranks and the least squares estimation method were used to estimate the parameters. The q-Weibull model, a generalization of other three models (Weibull, q-exponential and exponential) had the outstanding ability to clearly exhibits non-monotonic failure rates (bathtub curve and unimodal), describing the entire lifetime of two practical systems analyzed with a single set of parameters, that is impossible to the other distributions.*

Keywords: *reliability, q-distribution, failure rate, bathtub curve, sugarcane harvester*

1. INTRODUCTION

Reliability has developed to a high degree of refinement and qualification. It is the best quantitative measure of the integrity of a designed part, component, system or industrial operation (Kececioglu, 2002). A systemic approach based on mathematical models for the description of failure rates associated with equipment and components is necessary to ensure a safe operation and to maintain an operational process. There are many different statistical distributions that can be used to model life data.

The three-parameter Weibull distribution is one of the most common distributions applied in reliability analysis and has been widely used in many industrial applications, such as automotive, aerospace, military, nuclear power, electronics, electrical power, advertising, dental research and the mortality of mailing lists (Abernethy, 2006). This Weibull distribution is also indicated to evaluate the reliability of components that suffer wear-out failure and to determine the optimum replacement or repair interval (Mazhar *et al.*, 2007).

The survival probability density function (pdf) at time t described by Weibull (1951), can be written as:

$$f(t) = \frac{\beta}{\eta - t_0}\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta-1} \exp\left[-\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta}\right], \tag{1}$$

where t is the time-to-failure or lifetime, β is the shape parameter, $\eta - t_0$ (known as θ) is the scale parameter and t_0 is the location or minimum life parameter, with the constraints $\beta > 0$, $\eta - t_0 > 0$ and $t \geq t_0$. η and t_0 have the same unit of time t. β is dimensionless. When $\beta = 1$, the model recovers the two-parameter exponential distribution as a particular case of the Weibull model.

Despite the simplicity and the numerous applications, the Weibull distribution has some limitations. There are many generalizations or modifications based on the Weibull distribution that can better describe sample data of the most varied types of nature. Murthy *et al.* (2004) provide a taxonomy for Weibull models: multivariate extensions, stochastic models, linear or nonlinear transformation of time, etc. A short summary of generalizations of the Weibull distributions with two or more parameters and their characteristics along with some others distribution models was given by Pham and Lai (2007). A review of some discrete and continuous versions of the modifications of the Weibull distribution can also be found in the literature (Almalki and Nadarajah, 2014). The use of generalizations models in reliability analysis is since long, however, certain modifications of the traditional Weibull distribution had their originality questioned (see details in Nadarajah and Kotz (2005)).

The q-Weibull distribution is a generalization with four parameters of the Weibull distribution, which has three parameters. The Weibull distribution is based on the exponential function of a negative power-law. The q-Weibull distribution uses a generalization of the exponential function, called q-exponential, which presents the asymptotic behavior to a power-law and recovers, also as a particular case, an exponential one.

Studies have shown that the q-Weibull presented better performances in time-to-failure fit than the usual three-parameter Weibull distribution, q-exponential and exponential ones. The first application of the q-Weibull distribution in reliability analysis seems to have been presented by Picoli *et al.*, (2003). In a comparative study between q-Weibull, Weibull, q-exponential and exponential distributions, they verified that the distribution that well described the basketball baskets is the q-exponential. For cyclone victims and brand-name drugs by retail sales, the Weibull distribution gave a good adjustment. On the other hand, for length of highway, only q-Weibull distribution gives a satisfactory adjustment. A research of modeling time-to-failure due to the dielectric rupture of oxides in electronic devices provided a better quality fit with the q-Weibull distribution than that obtained by Weibull (Costa *et al.*, 2006). A similar study of the fit of time-to-failure data from a natural gas recovery plant was performed by Sartori *et al.* (2009) and also showed the superiority of q-Weibull due to the additional q parameter.

Besides that, five distinct patterns of failure rate were identified for the q-Weibull distribution, determined by well-defined ranges of shape parameters of the model. The usual Weibull model is not able to represent the entire life cycle of an asset with a single function since it only expresses monotonous failure rates. The same happens with the q-exponential and exponential distributions. Mathematical properties of the q-Weibull model which were explored and led up by Assis *et al.* (2013), opened this new field of research that had not been covered earlier in the literature. They showed that the q-Weibull model can also display, besides the monotonic curves, non-monotonic failure rate form: the well-known bathtub curve (U-shaped) and the unimodal shape. Each type of failure rate behavior has a specific range of value of shape parameters.

They continued with this study by applying a lifetime description of a robotic welding station used in a manufacturing process showing that for modeling bathtub curve, the original Weibull model requires three functions, one for each monotonous failure rate segment decreasing, constant and increasing. However, the generalized q-Weibull is capable to reproduce the entire bathtub curve (besides the unimodal one) with a unique function of a single set of parameters (Assis *et al.*, 2015). Depending on the range of the shape parameters, the q-Weibull also reproduces monotonic failure rates but influenced of the q-exponential cut off condition. A comparison with the q-exponential and exponential models was also performed and confirmed the superiority of q-Weibull model in fitting the data and in flexibility to describe failure rates.

It is important to note that the q-Weibull distribution has been connected with the dimensionless entropic parameter q in the context of Tsallis statistics. The Tsallis (1988) pioneering paper has introduced a generalization of the concept of entropy:

$$S_q = k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1}\ (q \in \mathbb{R}), \tag{2}$$

where k is a conventional positive constant, W is the total number of possible configurations and p_i is the associated probabilities. At the limit $q \to 1$, the Boltzmann-Gibbs statistics is recovered $S_1 = -k\sum_{i=1}^{W} p_i \ln p_i$ (see Tsallis (2001) for a recent review).

In order to apply the q-Weibull distribution in reliability modeling, some others important mathematical functions which were generalized in the context of nonextensive statistical mechanics need to be considered. Tsallis (1994) also defined the q-logarithm and its inverse, the q-exponential:

$$\ln_q(x) = \frac{x^{1-q}-1}{1-q}\ (x > 0), \tag{3}$$

$$\exp_q(x) = \begin{cases} [1+(1-q)x]^{(1/(1-q))}, & \text{if } (1+(1-q)x) > 0 \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

in which $x, q \in \mathbb{R}$. At the limit $q \to 1$, these functions recover the usual logarithm $\ln_1(x) = \ln(x)$ and exponential $\exp_1(x) = \exp(x)$. The cut off condition presented in Eq. (4) avoid negative and complex numbers, in order to achieve the probabilities. The q-exponential change from exponential behavior (with $q = 1$) for asymptotic power-law with large x and $q > 1$ presenting stretched tail. More details about these mathematical operations can be seen in Picoli *et al.* (2003) and Assis *et al.* (2013).

The purpose of this paper is to compare four reliability distributions: q-Weibull, Weibull, q-exponential and exponential. Two of them are generalizations of their originals. A practical application based on time-to-failure of a harvester sugarcane gives detailed results that confirm a previous study lead by Assis *et al*. (2015).

In the following section, we present the generalized versions q-Weibull and q-exponential distributions, presenting important mathematical equations, the failure rate function and the method of parameter estimation. The comparison of the models is considered in Section 3. Concluding remarks are given in the last section.

2. *Q*-DISTRIBUTIONS

The q-Weibull model is obtained from the Weibull model (Eq. (1)) by means of replacing the exponential function by the q-exponential (Costa *et al.*, 2006). Thus, the probability density function of q-Weibull, with $t \geq t_0$, can be written as:

$$f_q(t) = (2-q)\frac{\beta}{\eta - t_0}\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta-1} \exp_q\left[-\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta}\right]. \tag{5}$$

The constraint $q < 2$ and the factor $(2-q)$ are necessary to ensure the normalization of $f_q(t)$.

The following q-Weibull expressions was discussed and applied in Assis *et al.* (2013). The reliability function is defined by:

$$R_q(t) = \left\{\exp_q\left[-\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta}\right]\right\}^{2-q}. \tag{6}$$

The probability of failure $F_q(t)$ defines the cumulative fraction of parts that will fail by a time t:

$$F_q(t) = 1 - R_q(t) = 1 - \left\{\exp_q\left[-\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta}\right]\right\}^{2-q}. \tag{7}$$

It is important to note that Eq. (7) with parameters $q < 2$ and $\beta > 0$ is the generalization of the others three models presented in our study: the original Weibull (with $\beta > 0$ and $q = 1$), the q-exponential (with $q \neq 1$ and $\beta = 1$) and the original exponential (with $q = 1$ and $\beta = 1$ that also is a particular case of the original Weibull), as presented by Assis *et al.* (2015). It is also relevant to remark that for the generalized model of Weibull, when time value $t = \eta$ the cumulative failure for the characteristic life time $F_q(\eta)$ is not constant, depending on values of q parameter. For the three-parameter Weibull distribution ($q = 1$), $F_1(\eta) = 0.632$.

2.1 Failure rate

The failure rate function of the generalized q-Weibull distribution was introduced by Assis *et al.* (2013):

$$h_q(t) = \frac{f_q(t)}{R_q(t)} = \frac{(2-q)\beta}{\eta - t_0}\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta-1}\left\{\exp_q\left[-\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta}\right]\right\}^{q-1}. \tag{8}$$

The failure rate becomes the original Weibull when $q = 1$:

$$h(t) = \frac{\beta}{\eta - t_0}\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta-1}. \tag{9}$$

A total of five different behaviors of q-Weibull failure rate (Eq. (8)) can be described according to each range combination of shape parameters q and β: (a) monotonically decreasing ($1 < q < 2$ with $0 < \beta < 1$, $1 < q < 2$ with $\beta = 1$ and $q = 1$ with $0 < \beta < 1$); (b) monotonically increasing ($q < 1$ with $\beta = 1$, $q < 1$ with $\beta > 1$ and $q = 1$ with $\beta > 1$); (c) constant ($q = 1$ with $\beta = 1$); (d) bathtub curve ($q < 1$ with $0 < \beta < 1$); (e) unimodal ($1 < q < 2$ with $\beta > 1$). The other parameters η and t_0 do not change the shape type of failure rate. Assis *et al.* (2015) highlighted that the types (d) and (e) have a minimum and a maximum, respectively, and these behaviors cannot be achieved by the other distributions. The (c) type belongs to the exponential distribution and the failure rate is given by $h(t) = 1/(\eta - t_0)$.

2.2 Estimation of parameters

For the estimation of parameters, the sample data which are time-to-failure must be in ascending order. The median rank is the most popular approach of estimating the Y-axis plotting positions and regression analysis to fit the line. Weibull employed mean ranks in his paper but later, Johnson became recognized suggestions to use median ranks by means of Bernard`s approximation, an adjusted median rank (Johnson, 1951). Some examples using Bernard's approximation for the median rank were taken and demonstrated that is sufficiently accurate for plotting and estimating the parameters. It is also easier than interpolating in the tables for the adjusted median ranks that are not an integer value (Abernethy, 2006). So, an estimative of unreliability can be done:

$$\hat{F}_i = \frac{i - 0{,}3}{n + 0{,}4}, \tag{10}$$

where i is the failure order number which ranges from 1 to n and n is the sample size. Note that if two data points have the same time-to-failure on the X-axis, they are plotted at different median rank values on the Y-axis, each point gets its own individual vertical location. For each sampling time t_i, we have:

$$x_i = \ln(t_i - t_0), \tag{11}$$

$$y_i = \ln\left[-\ln_{q'}\left(1 - \widehat{F}_i\right)\right]. \tag{12}$$

Equation (7) can be described as $y = \beta x + b$, placing the sample data in a straight line by the change of variables x_i and y_i, represented by Eq. (11) and (12) and $b = -\beta\ln\left[(\eta - t_0)/(2-q)^{1/\beta}\right]$ (see Costa *et al.* (2006) and Assis *et at.* (2015) for details). Realize that there is a q-logarithm in y_i expression, denominated $q' = 1/(2-q)$, so the common procedure to calculate y_i for q-Weibull distribution must be changed, considering the generalized mathematical functions in the context of non-extensiveness mentioned above in Eq. (3) and (4). We obtain the graph of $\ln\left[-\ln_{q'}\left(1-\widehat{F}\right)\right]$ versus $\ln(t - t_0)$ since $\ln\left[-\ln_{q'}\left(1 - F_q(t)\right)\right] = \beta\ln(t - t_0) - \beta\ln\left[(\eta - t_0)/(2-q)^{1/\beta}\right]$, as described in Picoli *et al.* (2003).

The parameters of q-Weibull distribution (β, t_0, η and q) are estimated via the least squares estimation (LSE) method, maximizing the coefficient of determination, searching the parameters q and t_0 that return the maximum value of R^2, which represents the quality of the fit:

$$R^2 = 1 - \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}, \tag{13}$$

where the adjustment curve of the model $\hat{y}_i$ is $\ln\left[-\ln_{q'}\left(1 - F_q(t_i)\right)\right]$ and the mean $\bar{y}$ is $\sum y_i/n$. The constraints are $\beta > 0$, $\eta > t_0$, $\theta > 0$, $t_0 < t_{min}$ and $q < 2$. t_{min} is the lowest sample time. Note that the parameters of the original Weibull distribution can be obtained by imposing a constraint $q = 1$ on the four-parameter q-Weibull distribution. Equation (13) return $R^2 \leq 1$, including negative values.

The change of variables allows deducing the Eq. (14) to calculate the η parameter (Sartori *et al.*, 2009):

$$\eta = \exp\left(\frac{-b}{\beta}\right)(2-q)^{1/\beta} + t_0\,. \tag{14}$$

Other methods of parameter estimation were proposed. The maximum likelihood estimation (MLE) method was used in a detailed study by Jose and Naik (2009) and showed the properties of the q-Weibull distribution in applications to a data on cancer remission times. The results showed that the q-Weibull model has a better fit than Weibull, but they claimed the difficult to estimate the parameters due to the nonlinear set of equations. Other attempts with the MLE method were also performed and showed some difficulties with original Weibull distribution. It was cleared that the calculation is difficult and iterative for the Weibull parameters and so convergence may not always occur. It was also shown that it is satisfactory particularly for large samples over 500 failures (Abernethy, 2006). However, another way of estimating parameters has been presented using also the likelihood function, despite the difficulty to converge and it was applied with the fresh generalized q-Weibull model discussed here. To overcome this problem, Xu *et al*. (2017) proposed recently an adaptive hybrid artificial bee colony algorithm denominated AHABC. The parameter estimation procedure was applied to real reliability failure data and showed its effectiveness by producing a more accurate convergence.

Despite the good results presented by Xu *et al*. (2017) in dealing with nontrivial parameter estimation due to the intricate system of nonlinear equations, showing that the algorithm efficiently finds the optimal solution for the q-Weibull MLE problem, the maximum likelihood method was not used in our study.

3. APPLICATION TO A SUGARCANE HARVESTER

In this section, we apply the q-type distributions presented with their corresponding original versions, to analyze the time-to-failure from the historic industrial database of a sugarcane harvester. This machine has seven systems previously hierarchized: drive, electric, feeding, hydraulic, propulsion, straw extraction and transport of sugarcane. Particularly, we analyze the systems of the straw extraction and the drive, in which have respectively, 31 and 61 operation times, in hours. There is no censored data in the samples. All values of the samples were used to estimate the unreliability, according to the x_i and y_i variables and median ranks.

Table 1 shows the fitting parameters and the coefficient of determination R^2 for each system and models.

Figures 1 and 4 compares the four models by means of a graph of the variables y (Eq.(12)) versus x (Eq.(11)), for each time-to-failure t_i of both samples data. This representation shows how good the alignment of points to a straight line is. Besides being able to be assessed by the values of R^2 in Tab. 1, a visual inspection shows that q-Weibull fits better the data compared to the other models, especially in the tails, in both systems analyzed. The scales of the X-axis are different for each graph because the models have different values for the t_0 parameter. For the original models (Weibull and exponential), the scales of the Y-axis are equals, which is not the case of the generalized models due to the q-logarithm (Eq.(3)) present in the y variable and also by an additional parameter q.

Table 1. Fit parameters to q-Weibull, Weibull, q-exponential and exponential distributions for the analyzed systems.

System		q-Weibull	Weibull	q-exponential	Exponential
	β	0.81	1.15	1.00	1.00
	η (hour)	1,090	89	193	91
Straw extraction	t_0 (hour)	-0.37	-2.65	-1.79	-0.91
	q	-2.83	1.00	0.31	1.00
	R^2	0.9908	0.9754	0.9863	0.9702
	β	4.55	1.39	1.00	1.00
	η (hour)	132	300	909	332
Drive	t_0 (hour)	-122.11	1.24	8.26	9.35
	q	1.63	1.00	-0.16	1.00
	R^2	0.9800	0.9703	0.9492	0.9259

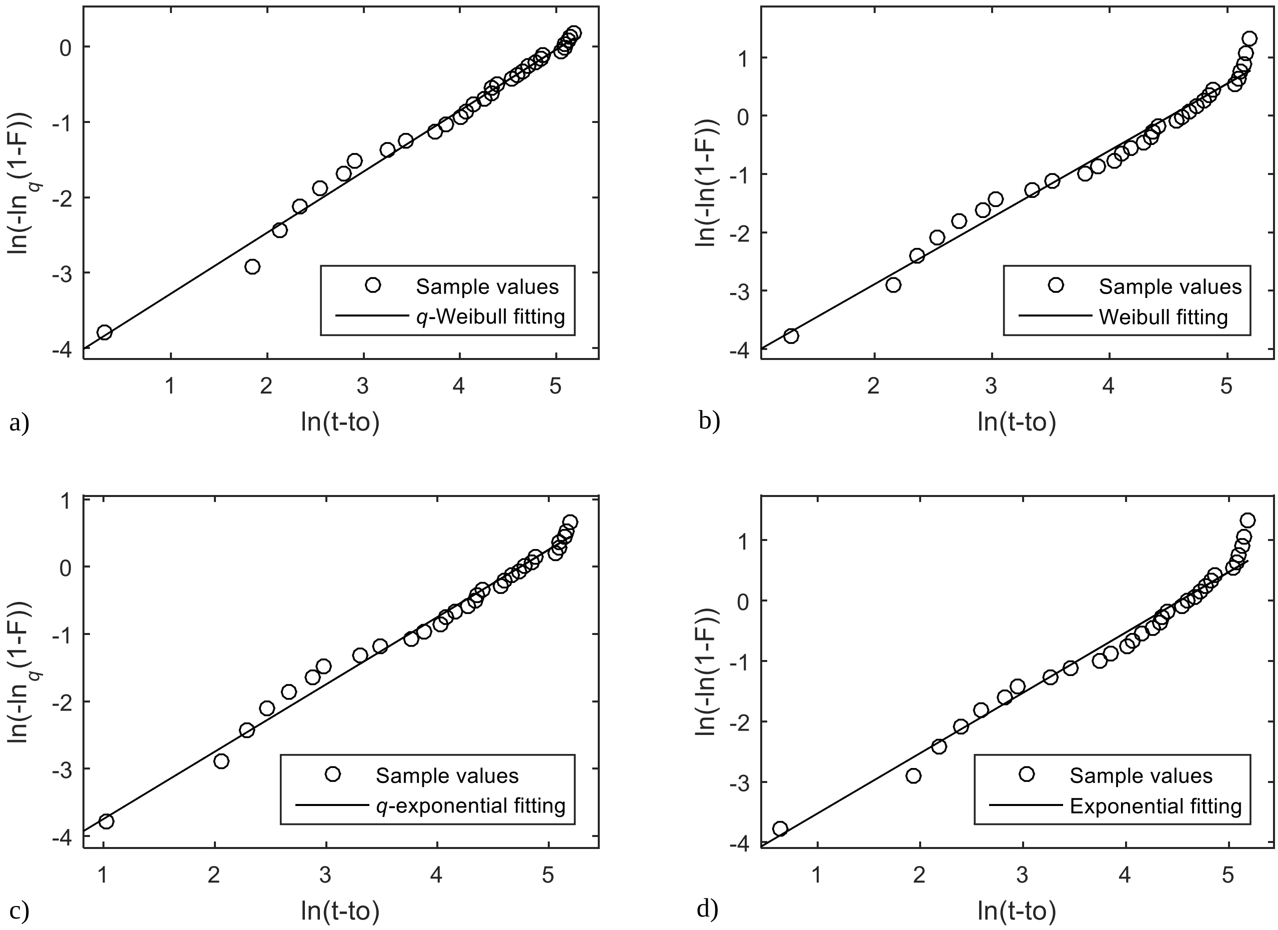

Figure 1. Fitting of the time-to-failure data (circles) of the straw extraction system and fitted curves (solid lines) of the models.

Figures 2 and 5 present the fittings for the reliability function $R_q(t)$, as a function of time-to-failure, for the four models and both samples data. It can be noted again that the q-Weibull model is able to fit better all the range of the data, while the others constantly diverges from experimental data, in both systems analyzed, especially in large time-to-failure. As the lifetime increases, the curves of the distributions move away and q-Weibull gets closer to the samples. This characteristic makes this model most appropriate for these sample data.

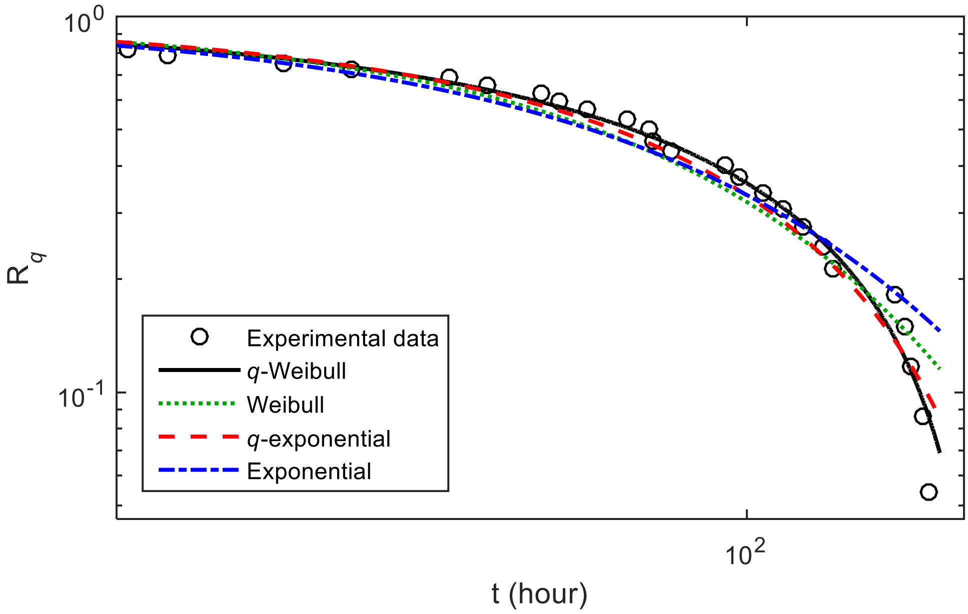

Figure 2. Fitting of the time-to-failure data (circles) of the straw extraction system and reliability curves for each model in a log-log plot.

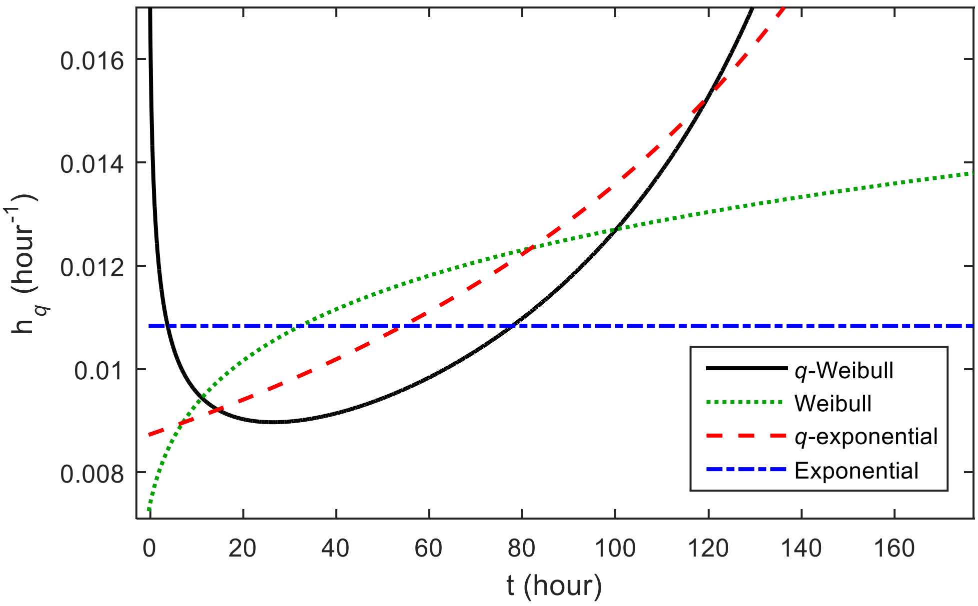

Figure 3. Failure rate curves for each model.

The shape of the failure rate $h_q(t)$ of the models is obtained through Eq. (8), by changing the range of parameters values. The unit of $h_q(t)$ is t^{-1}. For the systems analyzed, this parameter ranges and failure rate behaviors, as well as recovered functions, are displayed in Tab. 2. Figures 3 and 6 exhibit graphically these different failure rates, including non-monotonic forms. The first work recorded in the literature that reached such a representation of the bathtub curve and unimodal shape with the q-Weibull distribution is found in Assis *et al.* (2013), in which it provided resources for the results presented here.

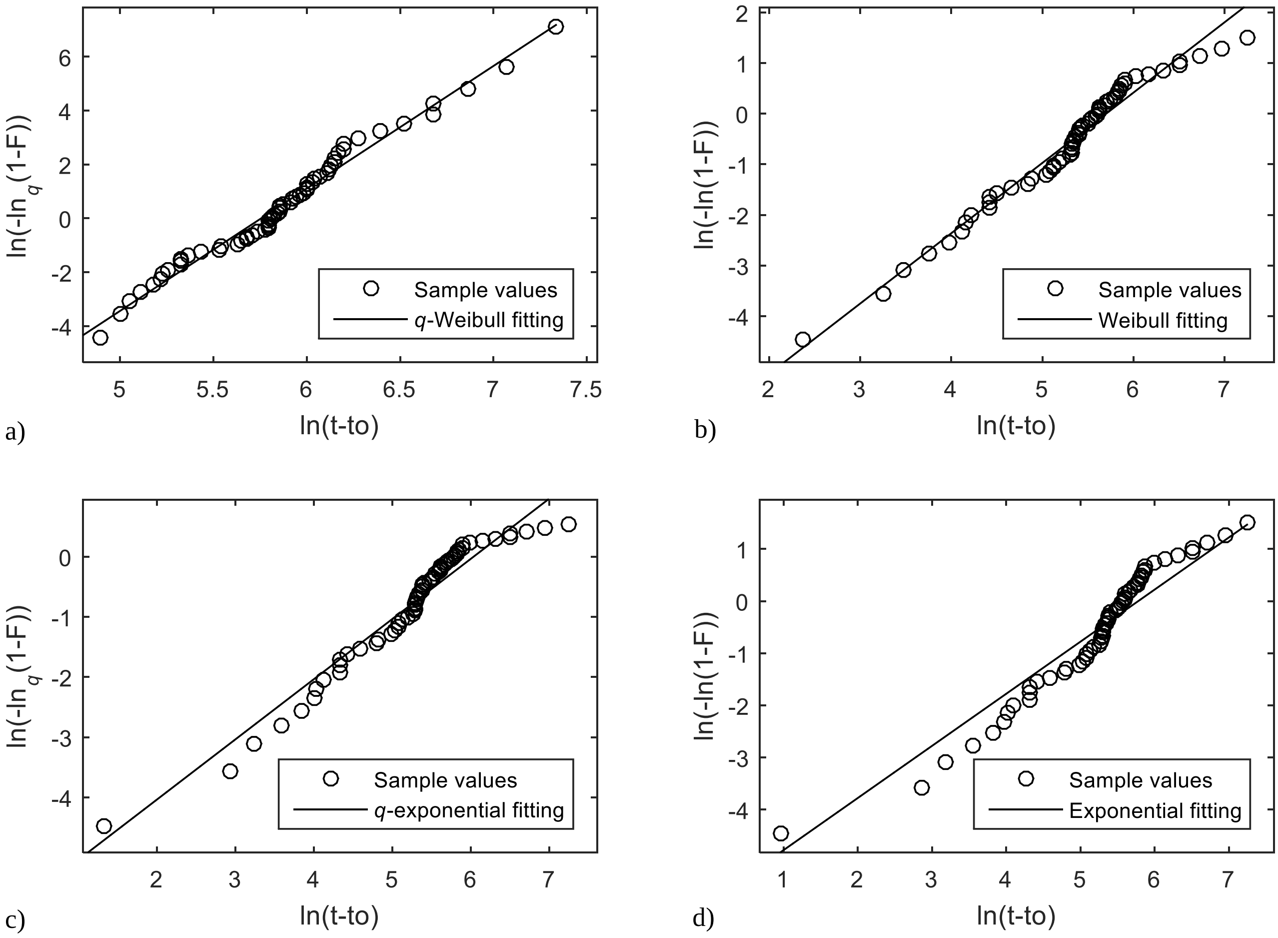

Figure 4. Fitting of the time-to-failure data (circles) of the drive system and fitted curves (solid lines) of the models.

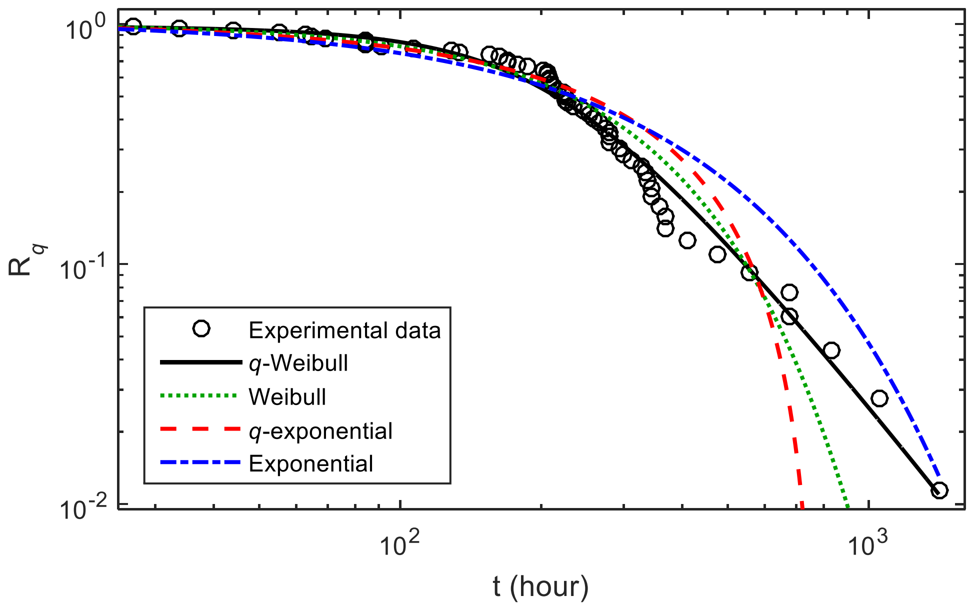

Figure 5. Fitting of the time-to-failure data (circles) of the drive system and reliability curves for each model in log-log plot.

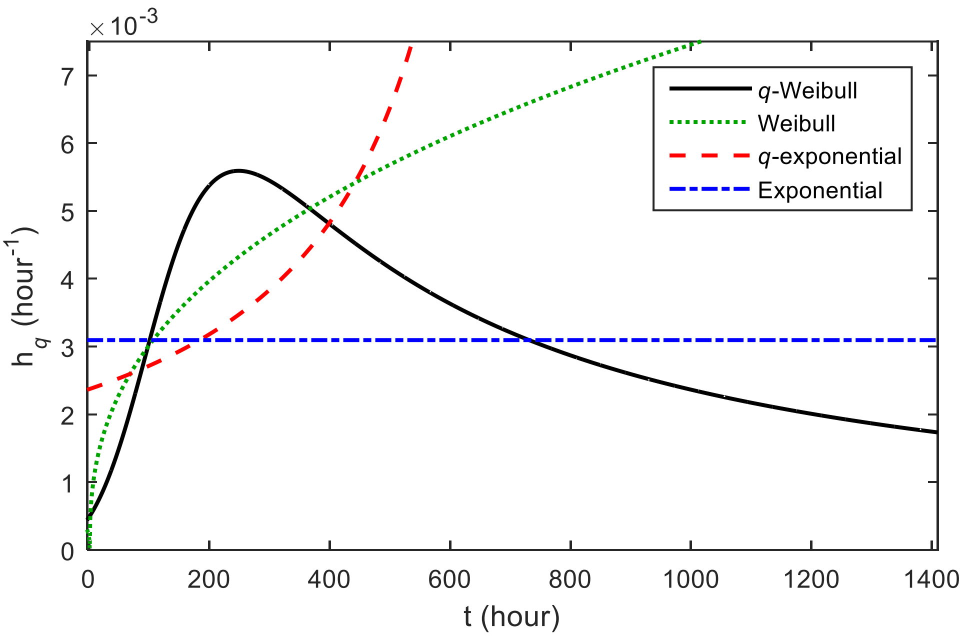

Figure 6. Failure rate curves for each model.

Table 2. Behavior of failure rate results and the models recovered for each system according to the range of the shape parameters q and β of the generalized q-Weibull.

System	q range	β range	Failure rate behavior	Model recovery
Straw extraction	$q < 1$	$0 < \beta < 1$	Bathtub curve	-
	$q = 1$	$\beta > 1$	Monotonous increasing	Weibull
	$q < 1$	$\beta = 1$	Monotonous increasing	q-exponential
	$q = 1$	$\beta = 1$	Constant	Exponential
Drive	$1 < q < 2$	$\beta > 1$	Unimodal	-
	$q = 1$	$\beta > 1$	Monotonous increasing	Weibull
	$q < 1$	$\beta = 1$	Monotonous increasing	q-exponential
	$q = 1$	$\beta = 1$	Constant	Exponential

In the following, we provide a brief analysis of each model. For a complete review, see Assis *et al*. (2015).

The exponential distribution has a constant failure rate that represents a single failure mode and nonrepairable items. The coefficient of determination $R^2 = 0.9259$ for drive system do not indicates a good fitness quality and so it shows that the constant failure rate does not correspond to the data, that include a variety of failure modes. The original Weibull distribution can reproduce monotonic failure rate, then the situation is closer to the data and can be modeled by this distribution. This model can describe failure rates monotonically decreasing, constant (as particular case with $\beta = 1$ that reduces to exponential) and monotonically increasing. For the same system mentioned in the exponential case, Weibull distribution presents a good coefficient of determination $R^2 = 0.9703$. The value of parameter $\beta > 1$ indicates a monotonically increasing failure rate behavior, however this result is a limitation of this model imposed to the data, as we shall see later. The q-exponential showed a slight improvement in fitting of the drive system data ($R^2 = 0.9492$) compared to its original. This model is not restricted to a constant failure rate, since is a generalization of exponential distribution. This presents a monotonic increasing ($q < 1$ and $\beta = 1$) in analysis of both systems. Finally, the q-Weibull distribution, the more flexible model considered in our analysis, is able to describe non-monotonic failure rates. The coefficient of determination $R^2 = 0.9800$ for the drive system and that obtained in the straw extraction system (greater than 0.99) confirms the best fit, that can also be visually verified in Figs. 1(a) and 2(a).

4. CONCLUSIONS

In summary, we have compared four reliability distributions to describe life data of a sugarcane harvester, in a specific mechanized cane cutting process. Two of these distributions are the generalized extensions of their ones. The generalization concerns a connection with the dimensionless parameter q, related to q-exponential function in the context of nonextensive statistical.

The approximation of the median ranks and the least squares estimation (LSE) method were used to estimate the parameters of the models, seeking the maximum value of the coefficient of determination R^2. The results show that the q-Weibull distribution performs better than the other distributions, showing to be the most suitable model for both analyzed systems. The q-Weibull model fits the sample data better and presents, therefore, an improvement to describe events, since the prediction of failures can be obtained with greater precision. Such an improvement was expected in fit of data due to the additional parameter q, but it is important to note the presence of q-exponential (and its inverse, the q-logarithm) in q-Weibull expressions. The q-Weibull showed be the only model with the ability to describe three distinct behaviors of failure with a single set of parameters: decreasing, constant, and increasing. These three phases are also known as infant mortality, useful life and wear out, respectively. When together and in this order, these phases form the non-monotonic behavior, known as bathtub curve (or U-shaped). The unimodal shape is also described by q-Weibull and has been presented by the drive system. The other distributions are unable to describe such types of curve, once is monotonically decreasing and monotonically increasing (Weibull and q-exponential distributions) or constant (exponential distribution). As it can be seen in Assis *et al*. (2015), for bathtub curve modeling, original Weibull model requires three functions, one for each failure rate segment decreasing, constant and increasing. These three functions indeed demand more effort and a larger set of parameters, while q-Weibull describes nonstop the entire bathtub curve maintaining the same set of four parameter values.

The advantage of being originated from a nonextensive statistical background makes the q-Weibull a promising alternative distribution for reliability modeling.

5. AKNOWLEDGEMENTS

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