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# AN INVERSE PROBLEM USING GREEN'S FUNCTIONS AND TFBGF METHOD TO IDENTIFY A MOVING HEAT SOURCE IN 3D HEAT CONDUCTION

**Ribeiro, S. S**

**Oliveira, G. C**

Federal University of Uberlândia

sidney.ribeiros@hotmail.com

gabriela\_costadeoliveira@hotmail.com

**Guimarães, G**

gguima@ufu.br

**Abstract.** Moving heat source are present in numerous problem practical in engineering. For example, machining process as the Gas tungsten arc welding (GTAW), laser welding, friction stirwleing process or milding problem. Moving heat source are also present in biological heating as the metabolism or in heat thermal treatment. All this case, the heat input identification is a complex task and represents an important factor in the optimization of the process. The aim of this work is to investigate both the temperature field as the heat flux delivered to a piece during a process with moving heat source.

**Keywords:** Inverse problem, analytical solution, moving heat source, Green's function

## 1. INTRODUCTION

Moving heat source are present in numerous problem practical in engineering. For example, machining process as cutting, milling or welding process. Moving heat source are also present in biological heating as the metabolism or in heat thermal treatment. All this case, the heat input identification is a complex task and represents an important factor in the optimization of the process.

Many investigators have studied welding heat flow problems, analytically, numerically, and experimentally. In the majority of these works, however, the values of the heat flux input is assumed to be known, taken from literature or determined by using calorimetric techniques. In fact, the heat flux that goes to the workpiece is unknown and its determination represents an inverse problem. In this sense, the use of techniques found in inverse heat conduction problems can represent an alternative way to obtain the heat flux that goes to the workpiece.

Moving heat source estimation without use of minimization least square, or optimization technique is the great advantages of the technique proposed here. The moving heat source can, then, be obtained directly from the temperature measured since the 3D transient analytical solution is obtained and the TFBGF (Fernandes, 2013) can be applied in that solution.

The temperature measurements are obtained using thermocouples at accessible regions of the workpiece surface while the theoretical temperatures are calculated from a 3D transient heat conduction thermal model with a moving heat source. The thermal model solution is obtained analytically (direct problem) (Ribeiro, 2015). The inverse problem, it means, the estimation of the moving heat source, uses the Transfer Function Based on Green's Function (TFBGF) method (Fernandes *et al.*, 2015). This method is based on Green's function and in the equivalence between thermal and dynamic systems. The technique is a simple approach without iterative processes, and therefore extremely fast. From the knowledge of both the temperature profile (hypothetical or experimental temperature far from the heat source) and of the transfer function it is possible to estimate the heat flux by an inverse procedure of the Fast Fourier Transform (IFFT). The TFBGF is, then, adapted to solve an inverse heat conduction problem with a moving heat source. Simulated and experimental test are used for estimating the heat source delivered to the workpiece.

## 2. GOVERNING EQUATION

The thermal problem can be described by the 3D heat conduction equation in the fixed  $x, y, z$  coordinate system, assuming constant thermal properties, as

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q_g}{k} \delta(x - vt) \delta(y - P_y) \delta(z - P_z) = \frac{1}{\alpha} \frac{\partial T}{\partial t} - u \frac{\partial T}{\partial x} \quad (1)$$

subjected to the first kind boundary conditions

$$\begin{aligned} T(0, y, z, t) &= T(L_1, y, z, t) = T_0 \\ T(x, y, z, t) &= T(x, L_2, z, t) = T_0 \end{aligned} \quad (2)$$

and the initial condition

$$T(x, y, z, 0) = T_0 \quad (3)$$

Machining, grinding, cutting, and sliding of surfaces, that has energy generated as a result of friction heating in a area with depth penetrarion, it means a volume heat source. In this sense, in this work, the heat source moving will be considered a point heat source of constant strength  $\delta(W)$ , releasing its energy continuously over time while moving along the  $x$  axis in the positive  $x$  direction with a constant velocity  $u$ , in a stationary medium that is initially at zero temperature.

After performing variables change using:

$$T(x, y, z, t) = W(x, y, z, t) \exp\left(\frac{ux}{2\alpha} - \frac{u^2 t}{4\alpha}\right) \quad (4)$$

In addition, in the solution of moving heat source problems, it is convenient to let the coordinate system move with the source Özişik (1993). This is achieved by introducing a new coordinate  $\xi$  defined by

$$\xi = x - ut \quad (5)$$

Thus, considering Eqs. (4) and (5) the governing equation is obtaned as

$$\frac{\partial^2 W}{\partial \xi^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} + \frac{q_g}{k} \delta(\xi - P_\xi) \delta(y - P_y) \delta(z - P_z) = \frac{1}{\alpha} \frac{\partial W}{\partial t} \quad (6)$$

subjected to the boundary conditions

$$W(0, y, z, t) = W(L_1, y, z, t) = 0 \quad (7)$$

$$W(\xi, 0, z, t) = W(\xi, L_2, z, t) = 0 \quad (8)$$

$$W(\xi, y, 0, t) = W(\xi, y, L_3, t) = 0 \quad (9)$$

and

$$k \frac{\partial W}{\partial z} \Big|_{z=L_3} + W(L_3, t) \left( h + \frac{ku}{2\alpha} \right) = hW_\infty e^{-\frac{uL_3}{2\alpha} - \frac{u^2 t}{4\alpha}} \quad (10)$$

where

$$h + \frac{ku}{2\alpha} = h_{eff} \quad (11)$$

The term  $h_{eff}$  in Eq.(11) is the effective heat convection coefficient Beck *et al.* (2010).

Equation (6) can then be solved by Green's function method Beck and McMasters (2004) as

$$W = \int_{\tau=0}^t \int_{\xi'=0}^{L_1} \int_{y'=0}^{L_2} \int_{z'=0}^{L_3} G^*(t - \tau) q_g(\tau) dz' dy' d\xi' d\tau \quad (12)$$

where

$$\begin{aligned} G^*(\xi, y, z, t | \xi', y', z', t - \tau) &= G(t - \tau) \delta(\xi' - P_\xi) \delta(y' - P_y) \delta(z' - P_z) \\ &\times e^{\frac{uP_\xi}{2\alpha} - \frac{u^2 \tau}{4\alpha}} \end{aligned} \quad (13)$$

and the Green's function for this problem can be obtained by Beck *et al.* (2010)

$$\begin{aligned}
 G(\xi, y, z, t | \xi', y', z', t - \tau) = & \frac{8}{L_1 L_2 L_3} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} e^{-\beta_m^2 u} e^{-\beta_n^2 u} e^{-\beta_p^2 u} \\
 & \times \text{sen}(\beta_m x) \text{sen}(\beta_m x') \text{sen}(\beta_n y) \\
 & \times \text{sen}(\beta_n y') \text{sen}(\beta_p z) \text{sen}(\beta_p z') \frac{\beta_p^2 + B^2}{\beta_p^2 + B^2 + B}
 \end{aligned} \tag{14}$$

Where  $\beta_m = \frac{m\pi}{L_1}$ ,  $\beta_n = \frac{n\pi}{L_2}$ ,  $-B = \beta_p \cot(\beta_p)$ ,  $B = \frac{hL_3}{k}$  e  $u = \alpha(t - \tau)$ .  
 Original temperature can then be recovered by Eq.(4).

$$T(x, y, z, t) = W(x, y, z, t) \exp\left(\frac{ux}{2\alpha} - \frac{u^2 t}{4\alpha}\right) \tag{15}$$

Figure (2) presents the temperatures generated by 3D analytical solution using a gaussian heat source presented in figure (1) at the point  $(0.02, 0.05, L_3)$ , using  $\alpha = 23.1 \times 10^{-6} m^2 s^{-1}$ ,  $k = 80.2 W m^{-1} K^{-1}$  and heat source speed  $u = 0.001 m s^{-1}$  at initial position  $(0.00, 0.05, L_3)$  for the parallelepiped with edges  $L_1 = L_2 = L_3 = 0.1 m$

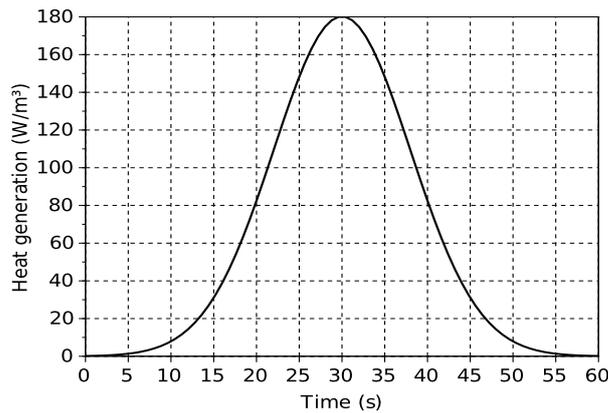


Figure 1. Gaussian heat source generation

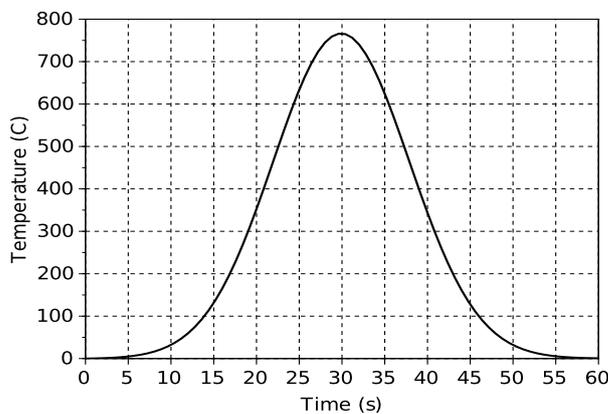


Figure 2. Temperatures for a gaussian distribution

## 2.1 THE TFBGF METHOD

For any dynamic system (Fig. 3), the relation between input  $X(t)$  and output  $Y(t)$  can be given by the convolution equation

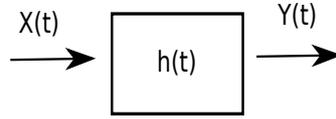


Figure 3. Diagram for dynamical systems

$$Y(t) = h(t) * X(t) = \int_{-\infty}^{+\infty} h(t - \tau)X(\tau)d\tau \quad (16)$$

That in Laplace domain is expressed by the multiplication

$$Y(x, s) = H(x, s) \cdot X(s). \quad (17)$$

In this sense, input  $X$  in domain  $s$  can be calculated by

$$X(s) = (1/H(x, s)) \cdot Y(x, s). \quad (18)$$

or in the time domain by the deconvolution

$$X(t) = \mathcal{L}^{-1}\{1/h(x, t)\} * Y(x, t). \quad (19)$$

Considering the heat source for input system and temperature field output, we can obtain the impulse response for this problem.

## 2.2 ANALYTICAL IMPULSE RESPONSE IDENTIFICATION

The proposed methodology for identification of the analytical impulse response is based on the theory of dynamical systems of one input and one output.

It can be observed that for any input  $X(t)$  and the output  $Y(t)$ , the transfer function,  $H(t)$ , remains the same.

Therefore, by comparing Eq.(13) and Eq.(16), and using transformation Eq. (4). In this case we use  $q_g(\tau) = \delta(\tau)$ . After the integral performing the Transfer function can then be calculated as

$$\begin{aligned} h(\xi, y, z, t) &= \frac{\alpha}{k} \frac{8}{L_1 L_2 L_3} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} e^{-(\beta_m^2 + \beta_n^2 + \beta_p^2)\alpha t} \\ &\quad \times \text{sen}(\beta_m \xi) \text{sen}(\beta_m P_\xi) \text{sen}(\beta_n y) \text{sen}(\beta_n P_y) \\ &\quad \times \text{sen}(\beta_p z) \text{sen}(\beta_p P_z) e^{\frac{u P_\xi}{2\alpha}} \left( \frac{\beta_p^2 + B^2}{\beta_p^2 + B^2 + B} \right) \end{aligned} \quad (20)$$

Thus, the heat source is obtained using

$$T(s) = q_g(s) \cdot H(s) \Rightarrow q_g(s) = \frac{T(s)}{H(s)} \quad (21)$$

The heat source moving can then be obtained in the original variable as

$$W(x, y, z, t) = T(x, y, z, t) \exp\left(\frac{-v_x x}{2\alpha} + \frac{v_x^2 t}{4\alpha}\right) \quad (22)$$

$$q_g(s) = \exp\left(\frac{v_x x}{2\alpha}\right) \frac{T(x, y, z, (s + b))}{H(s)} \quad (23)$$

where

$$a = \frac{-u x}{2\alpha} \quad (24)$$

and

$$b = \frac{u^2 t}{4\alpha} \quad (25)$$

Thus, the solution of the inverse problem (in original variable  $T$ ) is obtained by means of one of Eq. (22) in the Laplace domain or, in the time domain by Eq. (19) Fernandes *et al.* (2015).

In this case, the input, the heat source  $q_g(s)$ , that can be estimated from the impulse response  $H(s)$  and from the temperature measured at any position of the system, using the following expression

As mentioned, heat source estimation can be done in the Laplace domain or in time domain using the software Scilab with functions fast Fourier transform and inverse Fourier transform (*fft*) as the following expression

$$q_g(s) = (1/H(x, s)) \cdot W(x, s). \quad (26)$$

and

$$q_g(t) = \{1/h(x, t)\} * W(x, t). \quad (27)$$

### 3. HEAT SOURCE IDENTIFICATION

The three-dimensional is analyzed in this section. Temperature distributions for the direct problem are generated using the solution of Eq. (15) considering a moving gaussian heat source  $q_g(t)$  a shown in (Fig.1). Random errors are then added to these temperatures. The temperatures with error are then used in the inverse algorithm to reconstruct the imposed heat source. The simulated temperatures are calculated from the following equation

Using analytical solution, impulse response and hypothetical temperatures we can obtain the moving heat source term.

$$Y(t) = T(x, y, z, t) + \varepsilon \quad (28)$$

With the hypothetical temperatures (without added noise) calculated using Eq. (15) and the impulse response of the system (Fig. 4) calculated using Eq. (22) we obtain the temperature that will be used for estimations of the heat moving source using Eqs. (26 or 27).

Eqs. (15) or (16) can also be used to estimate the heat moving source using temperatures at each position  $x, y, z$  The respective temperature evolution for each position heat source is presented in Fig. (1 ).

Figure (4) presents the impulse response relative to the thermocouple position that will be used to estimate the moving heat source. Any par Temperature/heat transfer fucntion can be used to the estiamtion.

The impulse response presents the same behavior for the any case because only dependent of material used.

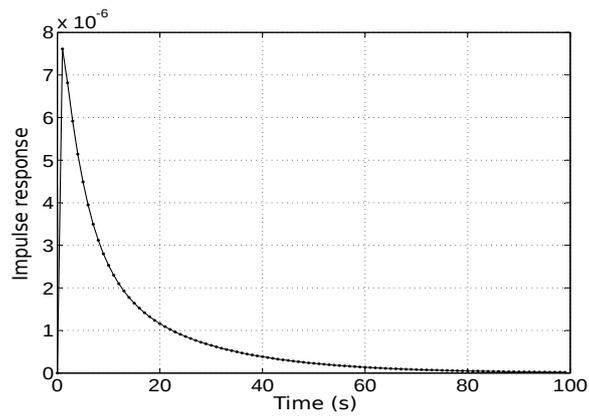


Figure 4. Impulse response

Figure (5) and presents the heat source estimated using  $(0.02, 0.04, L_3)$  position

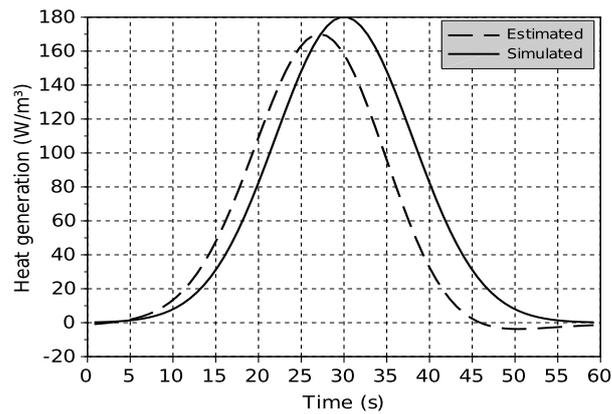


Figure 5. Estimated gaussian heat source compared with theoretical heat source

Since the heat moving heat source has been obtained, temperature can be calculated for the any region. In order to compare the results, Fig. (6) presents a comparison between simulated temperature and calculated temperatures using heat source estimated.

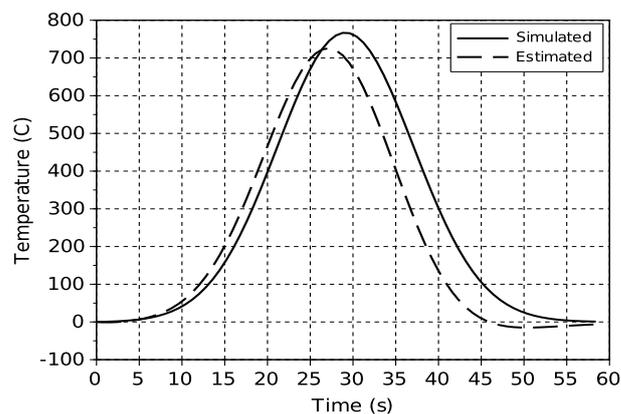


Figure 6. Temperatures for gaussian heat source

#### 4. CONCLUSION

The moving heat source has been obtained directly from the temperature measured since the 3D transient analytical solution is obtained and the TFBGF can be applied in that solution.

Use of the Transfer Function Based on Green's Function (TFBGF) method allows the estimation of heat source without iterative processes, and therefore extremely fast.

From the knowledge of both the temperature profile (hypothetical or experimental temperature far from the heat source) and of the transfer function it is possible to estimate the heat flux by an inverse procedure of the Fast Fourier Transform (IFFT) .

#### 5. ACKNOWLEDGEMENTS

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