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ANALYSIS THE TEMPERATURE OF THREE-DIMENSIONAL ANALYTICAL SOLUTION IN HEAT CONDUCTION TWO-LAYER

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Abstract. This paper presents a method of obtaining the three-dimensional (3D) analytic solution of the temperature for the multilayer heat transfer problem using Green Functions (FG), due to a transient three-dimensional thermal problem, subject to the condition of convexity on all faces. The advantage in using the FG method is that 3D problems becomes a simple multiplication of one-dimensional (1D) problems. This work not only presents the analytical solution but also its computational implementation, which allows us to better understand the physical problem. Obtaining the 3D analytical solution for the two-layer heat conduction problem in one direction requires more elaborate procedures than solving single-layer problems, both to fit the solution equation in terms of Green's functions and to obtain Two eigenvalues. A two-layer 3D problem in one direction, in perfect contact, the problem is referred to as $X33Y2C12Z33$ in (Haji-Sheikh, 2014). The temperature profile for the double-layer medium is obtained and verified the analytical solution through comparison with exact and numerical solutions of correlated and specific thermal problems.

Keywords: Analytical solution, multilayer, Green's functions, heat conduction

1. INTRODUCTION

Science is divided into three broad areas: human, biomedical, and exact, which are subdivided into two major groups, such as basic and applied sciences. As an example of the basic sciences is mathematics, physics and chemistry. Mathematics is a strong ally in the pursuit of desirable simplicity, security and confidence whenever any numerical solution or proposed computational computation can not be observed experimentally. Engineering, geosciences and astronomy can be seen as areas of application of these sciences (Oliveira, 2015).

Specifically, in the field of mechanical engineering among the existing phenomena, we study the heat transfer by conduction that occurs due to the temperature gradient in solid medium and that can be modeled mathematically by the diffusion equation.

It is proposed here to obtain the analytical solution for the heat conduction equation, which is given by a partial differential equation. Several methods can be used for this task, among them the Green Functions (FG) method is used since the boundary conditions vary over time, which immediately discards the method of separating variables.

One of the advantages in the use of integral solutions by FG is the possibility of constructing, without additional difficulties, multidimensional solutions from the one-dimensional Green functions. In this case, the versions of the 2D and 3D solution equations are absolutely equivalent to the one-dimensional equation and the GF can be obtained from products of 1D solutions in different directions (Oliveira, 2015).

The objective of the present work is the investigation and development of an analytical solution for heat conduction problems in multilayer media, or also called compounds, using the FG technique.

It is observed that the work not only presents the formulation, the development and the obtaining of the multi-layered analytical solution equation, but also its computational implementation, allowing a better physical understanding of the problem.

2. MATERIALS AND METHODS

The development of the dual-layer 3D analytical solution is presented below, which is represented by $X33Y2C12Z33$ in (Haji-Sheikh, 2014).

2.1 Three-dimensional transient thermal problem $X33Y2C12Z33$

The 3D heat conduction problem shown in Fig. 1 is a heat conduction problem whose all faces, except where the heat flow occurs, are subjected to a heat exchange by convection. Note that the problem consists of two layers in the direction of the axis y, whose thermophysical properties are different in each layer, delimited by $y = b$.
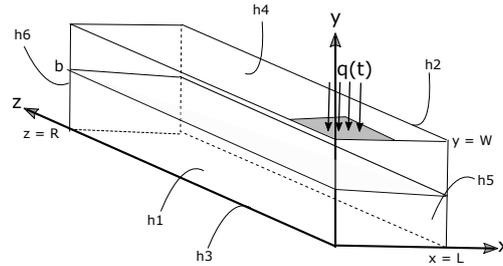

Figure 1. Problema $X33Y2C12Z33$

The problem represented by Fig. 1 with homogenized equations is given by:

$$\frac{\partial^2 \theta_1}{\partial x^2} + \frac{\partial^2 \theta_1}{\partial y^2} + \frac{\partial^2 \theta_1}{\partial z^2} = \frac{1}{\alpha_1} \frac{\partial \theta_1}{\partial t} \tag{1a}$$

$$\frac{\partial^2 \theta_2}{\partial x^2} + \frac{\partial^2 \theta_2}{\partial y^2} + \frac{\partial^2 \theta_2}{\partial z^2} + g(x, y, z, t) = \frac{1}{\alpha_2} \frac{\partial \theta_2}{\partial t} \tag{1b}$$

Subject to the boundary conditions on the x axis

$$-k_1 \frac{\partial \theta_1}{\partial x}\bigg|_{x=0} = -h_1 \theta_1; \qquad -k_1 \frac{\partial \theta_1}{\partial x}\bigg|_{x=L} = h_2 \theta_1 \tag{1c}$$

Subject to the boundary conditions on the y axis

$$-k_1 \frac{\partial \theta_1}{\partial y}\bigg|_{y=0} = -h_3 \theta_1; \qquad -k_2 \frac{\partial \theta_2}{\partial y}\bigg|_{y=W} - h_4 \theta_2 = 0 \tag{1d}$$

Subject to the boundary conditions on the z axis

$$-k_1 \frac{\partial \theta_1}{\partial z}\bigg|_{z=0} = -h_5 \theta_1; \qquad -k_1 \frac{\partial \theta_1}{\partial z}\bigg|_{z=R} = h_6 \theta_1 \tag{1e}$$

and to the continuity conditions

$$\theta_1\big|_{y=b} = \theta_2\big|_{y=b}; \qquad -k_1 \frac{\partial \theta_1}{\partial y}\bigg|_{y=b} = -k_2 \frac{\partial \theta_2}{\partial y}\bigg|_{y=b} \tag{1f}$$

And to the initial condition

$$\theta_1(x, y, z, 0) = \theta_2(x, y, z, 0) = F(x, y, z) - T_\infty \tag{1g}$$

The expression for the temperature in terms of the Green function is similar to that described for the case of one layer, but the functional form of the general solution and the Green function are obtained from more elaborate procedures. The general solution of the problem given by the equations (1a) - (1g) is shown in equation (1), assuming that the inhomogeneity $g(x, y, z, t) = q(t)\delta(y - W)$. This is due to the fact that the boundary conditions (heat flux) are removed by the procedure described in the superposition. Thus, the solution applies to the term for power generation.

The solution for temperature in each region i is given by

$$\theta_i(x,t) = \sum_{j=1}^{M} \left\{ \int_0^L \int_{y_j}^{y_{j+1}} \int_0^R G_{ij}(x,y,z,t|x',y',z',0)\theta_j(x',y',z',0)dx'dy'dz' \right.$$

$$\left. + \frac{\alpha_j}{k_j} \int_0^t \int_{L_1}^{L_2} \int_{R_1}^{R_2} g_j(x,y',z,\tau)G_{ij}(x,y,z,t|x',W,z',\tau)dx'dz'd\tau \right\}$$

(2)

Where the first term refers to the initial temperature term $(\theta(x,y,z,0))$ and the second term is related to the condition of heat flow boundary imposed to the arbitrary area $0 \leq L_1 \leq x \leq L_2 \leq L$ e $0 \leq R_1 \leq z \leq R_2 \leq R$ em $y = W$ as shown by Fig.(1).

It is observed that $y_j \leq y \leq y_{j+1}$, for $j = 1, 2, ..., M$, are the boundaries of each layer, and, $G_{ij}(x,y,z,t|x',y',z',\tau)$ is the Green function for multilayer problems.

If $M = 1$, the solution is for the case of a single layer within the range defined $0 \leq y \leq W$, where $y_1 = 0$ e $y_2 = W$, therefore the solution given by Eq. (2) is algebraically equal of a single layer.

If $M = 2$, defines two layers given by the following intervals $0 \ leqy \leq b$ e $b \ leqy \leq W$ where we have respectively layer 1 and layer 2, where $y_1 = 0$, $y2 = b$ and $y_3 = W$. Thus, the solutions have θ_1 e θ_2 defined respectively by Eqs. (3a) - (3b)

$$\theta_1(x,y,z,t) = \int_0^L \int_{y_1}^{y_2} \int_0^R G_{11}(x,y,z,t|x',y',z',0)\theta_1(x',y',z',0)dx'dy'dz'$$

$$+ \frac{\alpha_1}{k_1} \int_0^t \int_{L_1}^{L_2} \int_{R_1}^{R_2} g_1(x,y',z,\tau)\delta(y-W)G_{11}(x,y,z,t|x',W,z',\tau)dx'dz'd\tau$$

$$+ \int_0^L \int_{y_2}^{y_3} \int_0^R G_{12}(x,y,z,t|x',y',z',0)\theta_2(x',y',z',0)dx'dy'dz'$$

$$+ \frac{\alpha_2}{k_2} \int_0^t \int_{L_1}^{L_2} \int_{R_1}^{R_2} g_2(x,y',z,\tau)\delta(y-W)G_{12}(x,y,z,t|x',W,z',\tau)dx'dz'd\tau$$

(3a)

$$\theta_2(x,y,z,t) = \int_0^L \int_{y_1}^{y_2} \int_0^R G_{21}(x,y,z,t|x',y',z',0)\theta_1(x',y',z',0)dx'dy'dz'$$

$$+ \frac{\alpha_1}{k_1} \int_0^t \int_{L_1}^{L_2} \int_{R_1}^{R_2} g_1(x,y',z,\tau)\delta(y-W)G_{21}(x,y,z,t|x',W,z',\tau)dx'dz'd\tau$$

$$+ \int_0^L \int_{y_2}^{y_3} \int_0^R G_{22}(x,y,z,t|x',y',z',0)\theta_2(x',y',z',0)dx'dy'dz'$$

$$+ \frac{\alpha_2}{k_2} \int_0^t \int_{L_1}^{L_2} \int_{R_1}^{R_2} g_2(x,y',z,\tau)\delta(y-W)G_{22}(x,y,z,t|x',W,z',\tau)dx'dz'd\tau$$

(3b)

As the heat flow is applied to the surface, this implies that the second part of the equations (3a) - (3b) are void.

Obtain the Green function $G_{ij}(x,y,z,t|x',y',z'\tau)$ by observing the types of boundary conditions in the directions of x, y, being three independent one-dimensional problems. Then, we get $G_j(x,y,z,t|x',y',z'\tau)$ as a product of these Green functions, that is, $G_{ij}(x,y,z,t|x',y',z'\tau) = GX33GY2C13GZ33$.

In the x and y direction, we have the boundary conditions of type three, which means convection condition, and the one-dimensional Green function is easily found in (Cole *et al.*, 2010), but in y direction we have a unidimensional problem consisting of two layers submitted the convection condition in both directions, in this case requires obtaining the Grenn function.

In the x direction, we have

$$G_{X33}(x,t|x',\tau) = \frac{2}{L} \sum_{m=1}^{\infty} e^{-\alpha_m^2 \alpha(t-\tau)/L^2} \left[\alpha_m cos\left(\frac{\alpha_m(x)}{L}\right) + B_1 sen\left(\frac{\alpha_m(x)}{L}\right) \right]$$

$$\times \frac{\left[\alpha_m cos\left(\frac{\alpha_m(x')}{L}\right) + B_1 sen\left(\frac{\alpha_m(x')}{L}\right) \right]}{(\alpha_m^2 + B_1^2)\left[1 + \frac{B_2}{(\alpha_m^2 + B_2^2)}\right]} + B_1$$

(4)

where $\tan \alpha_m = \frac{\alpha_m(B_1+B_2)}{\alpha_m^2 - B_1 B_2}$, $B_1 = \frac{h_1 L}{k_1}$ e $B_2 = \frac{h_2 L}{k_1}$

In the z direction, we have

$$
G_{Z33}(z,t|z',\tau) = \frac{2}{R} \sum_{p=1}^{\infty} e^{-\gamma_p^2 \alpha(t-\tau)/R^2} \left[\gamma_p cos \left(\frac{\gamma_p(x)}{R} \right) + B_5 sen \left(\frac{\gamma_p(z)}{R} \right) \right]
$$
$$
\times \frac{\left[\gamma_p cos \left(\frac{\gamma_p(z')}{R} \right) + B_5 sen \left(\frac{\gamma_p(z')}{R} \right) \right]}{(\gamma_p^2 + B_5^2) \left[1 + \frac{B_6}{(\alpha_p^2 + B_6^2)} \right]} + B_5 \tag{5}
$$

where $\tan \gamma_p = \frac{\gamma_p(B_5 + B_6)}{\gamma_p^2 - B_5 B_6}$, $B_5 = \frac{h_5 R}{k_1}$ e $B_6 = \frac{h_6 R}{k_1}$.

In the y direction we have two layers and the Green multilayer function G_{ij} is given by (Haji-Sheikh and Beck, 2002)

$$
G_{ij}(y,t|y',\tau) = \sum_{n=1}^{\infty} e^{-\lambda_n^2(t-\tau)} \frac{1}{N_y} Y_{in}(y) Y_{jn}(y'), \tag{6}
$$

where Y_{in}, Y_{jn} are the eigenfunctions to be obtained, α_m, λ_n, and γ_p the eigenvalues that are obtained by means of transcendental equations that involve the tangent trigonometric function, the indices $m = 1, \cdots, M$, $n = 1, \cdots, N$ e $p = 1, \cdots, P$ define the number of interactions (eigenvalues) required for the convergence of series. Each B_i is the Biot number and N_y the norm.

The standard is defined as

$$
N_y = \sum_{j=1}^{M} \int_{y_j}^{y_{j+1}} \left[Y_{jn(y')} \right]^2 dy' \tag{7}
$$

From now on we will treat the problem in the y direction as being a one-dimensional problem and obtain the autofunctions for the two layers, ie, $Y_1 = Y_{1n}(y)$ e $Y_2 = Y_{2n}(y)$.

2.1.1 Obtaining the autofunctions Y_1 and Y_2

Assuming that the boundary conditions are homogeneous, it is proposed that the eigenfunctions are obtained by functions of independent variables in space and time, which will interest us only to determine the dependent function of y, which are the desired eigenfunctions get, so the method of separating variables will be used, so

$$
\theta_1(y,t) = Y_1(y)\Gamma_1(t) \tag{8a}
$$
$$
\theta_2(y,t) = Y_2(y)\Gamma_2(t) \tag{8b}
$$

Replacing Eqs. (8a) in Eqs. (1a) and Eqs. (8b) in Eqs. (1b) and dividing $Y_1(y)\Gamma_1(t)$ e $Y_2\Gamma_2(t)$ respectively:

$$
\frac{1}{Y_1} \frac{\partial^2 Y_1}{\partial y^2} = \frac{1}{\alpha_1 \Gamma_1} \frac{\partial \Gamma_1}{\partial t} \tag{9a}
$$

$$
\frac{1}{Y_2} \frac{\partial^2 Y_2}{\partial y^2} = \frac{1}{\alpha_2 \Gamma_2} \frac{\partial \Gamma_2}{\partial t} \tag{9b}
$$

and as first member of the equation (9) is independent of t and is equal to the second a member which is independent of y , it is concluded that both sides of the equation are independent y and t . Thus, each member of the equation must be a constant. Thus, rewriting the ordinary differential equations (EDOS) follows

$$
\frac{\partial^2 Y_1}{\partial y^2} + \phi^2 Y_1 = 0 \qquad\qquad \frac{\partial^2 Y_2}{\partial y^2} + \eta^2 Y_2 = 0 \tag{10}
$$

where

$$
\phi^2 = \frac{\lambda^2}{\alpha_1} \text{ and } \eta^2 = \frac{\lambda^2}{\alpha_2} \tag{11}
$$

The solutions to these EDOS are the eigenfunctions that one wishes to obtain:

$$
Y_1 = Acos(\phi y) + Bsen(\phi y) \tag{12a}
$$
$$
Y_2 = Ccos(\eta y) + Dsen(\eta y) \tag{12b}
$$

For the eigenfunctions Y_1 and Y_2 is necessary to obtain the coefficients of the equations (12a) and (12b). The Eqs. (12a) must satisfy the boundary condition in y = 0, is that

$$-k_1 \frac{\partial \theta_1}{\partial y}\Big|_{y=0} = -h_3\theta_1 \quad \Rightarrow \quad -k_1 \frac{\partial Y_1}{\partial y}\Big|_{y=0} = -h_3 Y_1 \quad \Rightarrow \quad \frac{\partial Y_1}{\partial y}\Big|_{y=0} = \frac{-h_3 Y_1}{-k_1} \tag{13}$$

Substituting the eigenfunction Y_1 Eq. (12a) in (13) and solving the expression, we obtain $B = 0$ and without loss of generality it is concluded that $A = 1$ (Özişik, 1993). Soon,

$$Y_1 \quad = \quad cos(\phi x) + \left(\frac{h_3}{k_1\phi}\right) sen(\gamma y) \tag{14}$$

Then you must satisfy the boundary conditions at $y = b$ which is given by

$$Y_1\big|_{y=b} = Y_2\big|_{y=b}; \qquad -k_1 \frac{\partial Y_1}{\partial y}\Big|_{y=b} = -k_2 \frac{\partial Y_2}{\partial y}\Big|_{y=b} \tag{15}$$

Replacing Y_1 and Y_2 in equation (15) follows

$$cos(\phi b) + \left(\frac{h_3}{k_1\phi}\right) sen(\phi b) - Ccos(\eta b) - Dsen(\eta b) = 0 \tag{16}$$

and

$$\left(\frac{k_1}{k_2}\right)\left(\frac{\gamma}{\eta}\right)\left[-sen(\phi b) + \left(\frac{h_3}{k_1\phi}\right)cos(\phi b)\right] + Csen(\eta b) - Dcos(\eta b) = 0 \tag{17}$$

The boundary condition at $y = W$ is defined as

$$-k_2 \frac{\partial \theta_2}{\partial y}\Big|_{y=W} = -h_4\theta_2 \quad \Rightarrow \quad -k_2 \frac{\partial Y_2}{\partial y}\Big|_{y=W} - h_4 Y_2 = 0 \tag{18}$$

Ousting Y_2 the equation (18) follows

$$-k_2[-C\eta sen(\eta W) + D\eta cos(\eta W)] - h_4[Ccos(\eta W) + Dsen(\eta W)] = 0 \tag{19}$$

Applied all boundary conditions, the equations (16), (17) and (19) in matrix form is given by

$$\begin{bmatrix} cos(\phi b) + Hsen(\phi b) & cos(\eta b) & sen(\eta b) \\ -Ksen(\eta b) + KHcos(\eta b) & -sen(\eta b) & cos(\eta b) \\ 0 & k_2\eta sen(\eta W) & -k_2\eta cos(\eta W) \\ & -h_4cos(\eta W) & -h_4sen(\eta W) \end{bmatrix} \begin{bmatrix} 1 \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{20}$$

where

$$K = \left(\frac{k_1}{k_2}\right)\left(\frac{\phi}{\eta}\right) \qquad H = \left(\frac{h_3}{k_1\phi}\right) \tag{21}$$

Solving the linear system given by the equation (20) coefficients C and D are determined

$$C = cos(\eta b)[cos(\phi b) + Hsen(\phi b)] + Ksen(\eta b)sen(\phi b) - KHsen(\eta b)cos(\phi b) \tag{22}$$

$$D = sen(\eta b)[cos(\phi b) + Hsen(\phi b)] - Kcos(\eta b)sen(\phi b) + KHcos(\eta b)cos(\phi b) \tag{23}$$

Substituting the eigenfunctions Y_1 and Y_2, the standard equation (7) is defined by:

$$N_y = \underbrace{\int_0^b [cos(\phi y') + \left(\frac{h_3}{k_1\phi}\right) sen(\phi y')]^2 dy'}_{1} + \underbrace{\int_b^W [Ccos(\eta y') + Dsen(\eta y')]^2 dy'}_{2} \tag{24}$$

Solving the integrals we have, where $H = \frac{h_3}{k_1 \phi}$

$$
\begin{aligned}
\text{Integral 1} &= \int_0^b [cos(\phi y') + H sen(\phi y')]^2 dy' \\
&= \int_0^b cos(\phi y')^2 + 2[cos(\phi y')H sen(\phi y')] + H^2 sen(\phi y')^2 dy' \\
&= \underbrace{\int_0^b cos(\phi y')^2 dy'}_{A} + \underbrace{2H \int_0^b cos(\phi y')sen(\phi y')dy'}_{B} + \underbrace{H^2 \int_0^b sen(\phi y')dy'}_{C}
\end{aligned}
\tag{25}
$$

Therefore, Integral I = Integral A + Integral B + Integral C.

$$
\begin{aligned}
\text{Integral 2} &= \int_b^W [Ccos(\eta y') + Dsen(\eta y')]^2 dy' \\
&= \int_b^W C^2 cos^2(\eta y') + 2CDcos(\eta y')sen(\eta y') + D^2 sen^2(\eta y')dy' \\
&= C^2 \int_b^W cos^2(\eta y')dy' + 2CD \int_b^W cos(\eta y')sen(\eta y')dy' + D^2 \int_b^W sen^2(\eta y')dy' \\
&= \frac{C^2}{4\eta}(2\eta(W-b) - sen(2b\eta) + sen(2L\eta)) + \frac{2CD}{4\eta}(cos(2b\eta) - cos(2L\eta)) \\
&\quad + \frac{D^2}{4\eta}(2\eta(W-b) + sen(2b\eta) - sen(2W\eta))
\end{aligned}
\tag{26}
$$

Therefore, the standard is defined by:

$$
N_y = \text{Integral 1} + \text{Integral 2}
\tag{27}
$$

It is observed that for obtaining the computational solution $X33Y2C12Z33$ obtain the eigenvalues needed. The eigenvalues for the thermal problem $X33Y2C12Z33$ should be obtained by a numerical method, because it is a transcendental equation. The equation for calculating the eigenvalue is obtained by the boundary conditions of continuity is at $y = W$, that is, writing the coefficients of the matrix has been

$$
\begin{vmatrix}
cos(\phi b) + H sen(\phi b) & cos(\eta b) & sen(\eta b) \\
-K sen(\eta b) + KH cos(\eta b) & -sen(\eta b) & cos(\eta b) \\
0 & k_2 \eta sen(\eta W) & -k_2 \eta cos(\eta W) \\
& -h_4 cos(\eta W) & -h_4 sen(\eta W)
\end{vmatrix} = 0
\tag{28}
$$

And calculating its determinant, we obtain the transcendental equation

$$
\frac{[\eta(b-W)]tan[\eta(b-W)] - \frac{h_1 b}{k_1}}{\frac{h_1 b}{k_1}tan[\eta(b-W)] + [\eta(b-W)]} = -K \frac{\phi b tan(\phi b) - \frac{h_1 b}{k_1}}{\frac{h_1 b}{k_1} + \phi b}
\tag{29}
$$

The solution of equation (29) can be obtained by applying various mathematical methods. (Beck, 1992) present solutions to the transcendental equation based on asymptotic approximations. Therefore, the solution temperature, is defined by

$$\theta_1(x,y,z,t) =$$

$$\frac{4}{N_y}\theta_1\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}e^{-\left(\frac{\alpha_m^2}{L^2}+\lambda_n^2+\frac{\gamma_p^2}{R^2}\right)\alpha(t)}\times\frac{\left[\alpha_m cos\left(\frac{\alpha_m x}{L}\right)+B_1 sen\left(\frac{\alpha_m x}{L}\right)\right]}{(\alpha_m^2+B_1^2)\left[1+\frac{B_2}{\alpha_m^2+B_2^2}\right]+B_1}$$

$$\times\left(cos(\phi y)+\frac{h_3}{k_1\phi}sen(\phi y)\right)\times\frac{\left[\gamma_p cos\left(\frac{\gamma_p z}{R}\right)+B_5 sen\left(\frac{\gamma_p z}{R}\right)\right]}{(\gamma_p^2+B_5^2)\left[1+\frac{B_6}{\gamma_p^2+B_6^2}\right]+B_5}$$

$$\times\frac{1}{\alpha_m\gamma_p}[\alpha_m sen\alpha_m-B_1(cos\alpha_m-1)]\times\left[\frac{sen(\phi b)}{\phi}+\frac{h_3}{k_1\phi^2}(\phi-cos(\phi b))\right]$$

$$\times[\gamma_p sen\gamma_p-B_5(cos\gamma_p-1)]$$

$$+\frac{4}{N_y}\theta_2\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}e^{-\left(\frac{\alpha_m^2}{L^2}+\lambda_n^2+\frac{\gamma_p^2}{R^2}\right)\alpha(t)}\times\frac{\left[\alpha_m cos\left(\frac{\alpha_m x}{L}\right)+B_1 sen\left(\frac{\alpha_m x}{L}\right)\right]}{(\alpha_m^2+B_1^2)\left[1+\frac{B_2}{\alpha_m^2+B_2^2}\right]+B_1}$$

$$\times\left(cos(\phi y)+\frac{h_3}{k_1\phi}sen(\phi y)\right)\times\frac{\left[\gamma_p cos\left(\frac{\gamma_p z}{R}\right)+B_5 sen\left(\frac{\gamma_p z}{R}\right)\right]}{(\gamma_p^2+B_5^2)\left[1+\frac{B_6}{\gamma_p^2+B_6^2}\right]+B_5}$$

$$\times\frac{1}{\alpha_m\gamma_p}[\alpha_m sen\alpha_m-B_1(cos\alpha_m-1)]$$

$$\times\left[\frac{-Csen(\eta b)+Dcos(\eta b)+Csen(\eta W)-Dcos(\eta W)}{\eta}\right] \qquad (30)$$

$$\times[\gamma_p sen\gamma_p-B_5(cos\gamma_p-1)]$$

$$+\frac{\alpha_2}{k_2}\frac{4}{N_y}\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}e^{-\left(\frac{\alpha_m^2}{L^2}+\lambda_n^2+\frac{\gamma_p^2}{R^2}\right)\alpha(t-\tau)}$$

$$\times\frac{\left[\alpha_m cos\left(\frac{\alpha_m x'}{L}\right)+B_1 sen\left(\frac{\alpha_m x'}{L}\right)\right]}{(\alpha_m^2+B_1^2)\left[1+\frac{B_2}{\alpha_m^2+B_2^2}\right]+B_1}\times\left(cos(\phi y)+\frac{h_3}{k_1\phi}sen(\phi y)\right)$$

$$\times\left(cos(\phi W)+\frac{h_3}{k_1\phi}sen(\phi W)\right)\times\frac{\left[\gamma_p cos\left(\frac{\gamma_p z'}{R}\right)+B_5 sen\left(\frac{\gamma_p z'}{R}\right)\right]}{(\gamma_p^2+B_5^2)\left[1+\frac{B_6}{\gamma_p^2+B_6^2}\right]+B_5}$$

$$\times\left[L\left[sen\left(\frac{\alpha_m L_2}{L}\right)-sen\left(\frac{\alpha_m L_1}{L}\right)\right]-\frac{B_1 L}{\alpha_m}\left[cos\left(\frac{\alpha_m L_2}{L}\right)-cos\left(\frac{\alpha_m L_1}{L}\right)\right]\right]$$

$$\times\left[R\left[sen\left(\frac{\gamma_p R_2}{R}\right)-sen\left(\frac{\gamma_p R_1}{R}\right)\right]-\frac{B_5 R}{\gamma_p}\left[cos\left(\frac{\gamma_p R_2}{R}\right)-cos\left(\frac{\gamma_p R_1}{R}\right)\right]\right]$$

$$\times\int_0^t\left[q(\tau)e^{\left(\frac{\alpha_m^2}{L^2}+\lambda_n^2+\frac{\gamma_p^2}{R^2}\right)\alpha(t)}\right]d\tau$$

$$\theta_2(x,y,z,t) =$$

$$\frac{4}{N_y}\theta_1 \sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-\left(\frac{\alpha_m^2}{L^2}+\lambda_n^2+\frac{\gamma_p^2}{R^2}\right)\alpha(t)} \times \frac{\left[\alpha_m cos\left(\frac{\alpha_m x}{L}\right) + B_1 sen\left(\frac{\alpha_m x}{L}\right)\right]}{(\alpha_m^2 + B_1^2)\left[1 + \frac{B_2}{\alpha_m^2+B_2^2}\right] + B_1}$$

$$\times [C cos(\eta y) + D sen(\eta y)] \times \frac{\left[\gamma_p cos\left(\frac{\gamma_p z}{R}\right) + B_5 sen\left(\frac{\gamma_p z}{R}\right)\right]}{(\gamma_p^2 + B_5^2)\left[1 + \frac{B_6}{\gamma_p^2+B_6^2}\right] + B_5}$$

$$\times \frac{1}{\alpha_m\gamma_p}[\alpha_m sen\alpha_m - B_1(cos\alpha_m - 1)] \times \left[\frac{sen(\phi b)}{\phi} + \frac{h_3}{k_1\phi^2}(\phi - cos(\phi b))\right]$$

$$\times [\gamma_p sen\gamma_p - B_5(cos\gamma_p - 1)]$$

$$+ \frac{4}{N_y}\theta_2 \sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-\left(\frac{\alpha_m^2}{L^2}+\lambda_n^2+\frac{\gamma_p^2}{R^2}\right)\alpha(t)} \times \frac{\left[\alpha_m cos\left(\frac{\alpha_m x}{L}\right) + B_1 sen\left(\frac{\alpha_m x}{L}\right)\right]}{(\alpha_m^2 + B_1^2)\left[1 + \frac{B_2}{\alpha_m^2+B_2^2}\right] + B_1}$$

$$\times [C cos(\eta y) + D sen(\eta y)] \times \frac{\left[\gamma_p cos\left(\frac{\gamma_p z}{R}\right) + B_5 sen\left(\frac{\gamma_p z}{R}\right)\right]}{(\gamma_p^2 + B_5^2)\left[1 + \frac{B_6}{\gamma_p^2+B_6^2}\right] + B_5}$$

$$\times \frac{1}{\alpha_m\gamma_p}[\alpha_m sen\alpha_m - B_1(cos\alpha_m - 1)]$$

$$\times \left[\frac{-C sen(\eta b) + D cos(\eta b) + C sen(\eta W) - D cos(\eta W)}{\eta}\right]$$

$$\times [\gamma_p sen\gamma_p - B_5(cos\gamma_p - 1)]$$

(31)

$$+ \frac{\alpha_2}{k_2}q_2(x,z,t)\frac{4}{LR}\frac{1}{N_y}\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-\left(\frac{\alpha_m^2}{L^2}+\lambda_n^2+\frac{\gamma_p^2}{R^2}\right)\alpha(t-\tau)}$$

$$\times \frac{\left[\alpha_m cos\left(\frac{\alpha_m x}{L}\right) + B_1 sen\left(\frac{\alpha_m x}{L}\right)\right]}{(\alpha_m^2 + B_1^2)\left[1 + \frac{B_2}{\alpha_m^2+B_2^2}\right] + B_1} \times [C cos(\eta y) + D sen(\eta y)]$$

$$\times \left(cos(\phi W) + \frac{h_3}{k_1\phi}sen(\phi W)\right) \times \frac{\left[\gamma_p cos\left(\frac{\gamma_p z}{R}\right) + B_5 sen\left(\frac{\gamma_p z}{R}\right)\right]}{(\gamma_p^2 + B_5^2)\left[1 + \frac{B_6}{\gamma_p^2+B_6^2}\right] + B_5}$$

$$\times \left[L\left[sen\left(\frac{\alpha_m L_2}{L}\right) - sen\left(\frac{\alpha_m L_1}{L}\right)\right] - \frac{B_1 L}{\alpha_m}\left[cos\left(\frac{\alpha_m L_2}{L}\right) - cos\left(\frac{\alpha_m L_1}{L}\right)\right]\right]$$

$$\times \left[R\left[sen\left(\frac{\gamma_p R_2}{R}\right) - sen\left(\frac{\gamma_p R_1}{R}\right)\right] - \frac{B_5 R}{\gamma_p}\left[cos\left(\frac{\gamma_p R_2}{R}\right) - cos\left(\frac{\gamma_p R_1}{R}\right)\right]\right]$$

$$\times \int_0^t \left[q(\tau)e^{\left(\frac{\alpha_m^2}{L^2}+\lambda_n^2+\frac{\gamma_p^2}{R^2}\right)\alpha(t)}\right]d\tau$$

3. RESULTS AND DISCUSSION

3.1 Checking the 3D solution X33Y3C13Z33

Solutions verification is necessary to ensure the consistency of analytically constructed solutions or numeric codes. As seen, the double-layered solution $X33Y2C13Z33$ is represented by Fig. (1) where the geometry analyzed is composed of different materials. To make the intrinsic verification of the solution, both layers are considered to have the same thermophysical properties. In this case, the thermal properties used are $k_1 = k_2 = 24\ [W/mK]$, $\alpha_1 = \alpha_2 = 7.0868e-06\ [m^2/s]$, heat flux prescribed equal $q = 1 \times 10^5\ [W/m^2]$, $T_0 = 25\ [^oC]$, $T_{inf} = 30\ [^oC]$, $h = 20$, $L = 1 \times 10^{-2}$, $W = 1 \times 10^{-2}$, $R = 10 \times 10^{-2}$ and $b = W/2$.

Three graphs showing the thermal profile of the solutions and the absolute difference between them are analyzed, analyzing the solutions in the respective points, y = W, y = b and y = 0.
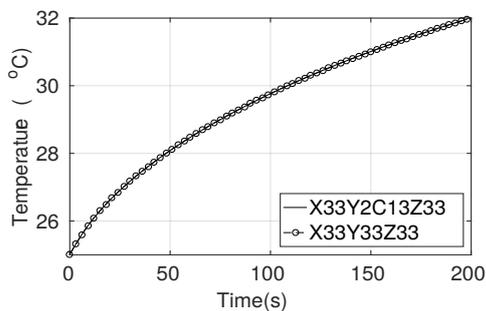
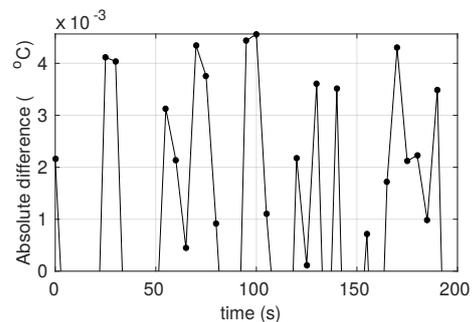It is observed that the maximum error obtained is in the third decimal place.
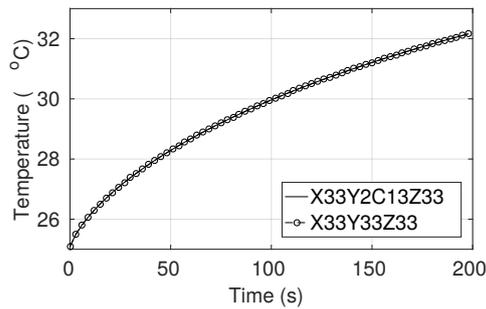
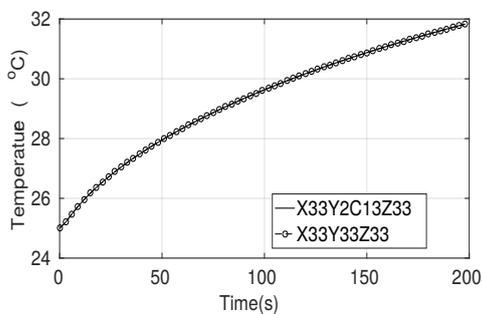(a) Checking solutions X33Y33Z33 and X33Y2C13Z33 at the point y=W.
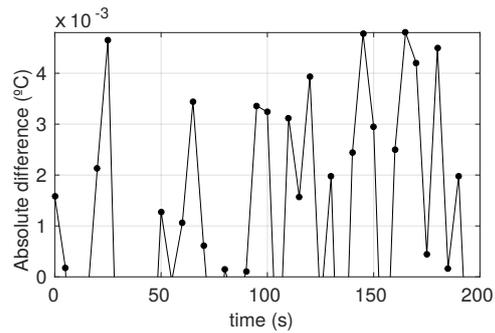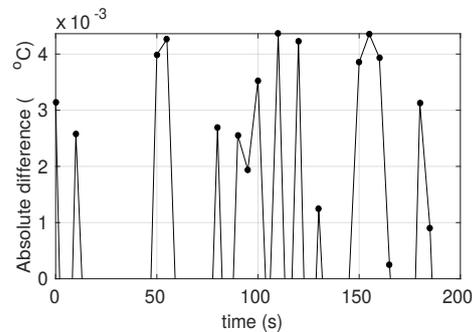

(b) Absolute difference y=W.

(c) Checking solutions X33Y33Z33 and X33Y2C13Z33 at the point y=b.

(d) Absolute difference y=b.

(e) Checking solutions X33Y33Z33 and X33Y2C13Z33 at the point y=0.

(f) Absolute difference y=0.

4. CONCLUSIONS

Obtained analytical solution for the temperature problem multilayer heat transfer $X33Y2C12Z33$, furthermore, if the multi-validated analytical solution, in which case it is considered the same thermophysical properties with the thermal problem $X33$ already verified.

5. ACKNOWLEDGEMENTS

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