24th ABCM International Congress of Mechanical Engineering
December 3-8, 2017, Curitiba, PR, Brazil

# COBEM-2017-0242
# ANALYTICAL SOLUTION FOR THE TRANSIENT HEAT TRANSFER PROBLEM WITH VARIABLE SOURCE TERM IN A NUCLEAR FUEL ROD

**Filipe Alcântara Soares**
**Priscila Pires Araújo**
**Francesco Scofano Neto**
**Rodrigo Otávio de Castro Guedes**
Department of Mechanical Engineering, Instituto Militar de Engenharia, RJ, Brazil
alcantara_filipe@hotmail.com, priscila.pires.engenergia@gmail.com, scofano@ime.eb.br, guedes@ime.eb.br

***Abstract.*** *This work presents a fully analytical solution for the transient conduction heat transfer with variable source term in fuel rods of nuclear reactors by employing the Classical Integral Transform Technique. In this contribution, two forms of source terms are considered: a time dependent one and a more realistic situation in which this parameter is both space and time dependent. These functions for the heat source are compared with others variable source terms found in the literature. Special attention is given to the prediction of the critical time which is the time required for the complete meltdown of the cladding material. The influence of the Biot number in the determination of the critical time is approached and it was seen that the Biot can play an important role in order to increase the critical time of the cladding. Furthermore, the developed solution showed a fast convergence rate.*

***Keywords:*** *Transient Thermal Analysis, Nuclear Fuel Rod, Variable Heat Source Term, Critical Time, Classical Integral Transform Technique.*

## 1. INTRODUCTION

A nuclear reactor is composed of fissile fuel rods where a nuclear fission reaction is processed, which is defined as the division of the core of a heavy atom into two smaller ones when struck by a neutron. As far as the fuel rod cooling process is concerned, after the collision, a large quantity of energy is released inside of the fuel rod and it is transferred by conduction to the surface of the element, and from there to the cladding material, where heat is transferred by convection to the coolant (Eskandari *et al*., 2012).

Fuel rods are the source of energy in nuclear reactors and are usually made out of small uranium-235 pellets within a cladding. A nuclear accident may happen due to the inadequate cooling of the rods, up to the point that the cladding starts melting (An *et al*., 2014).

According to Bhattacharya *et al.* (2001), the neutron flux being constant, the volumetric heat source term remains constant, and it can be evaluated by the knowledge of the fissionable cores number and the neutron flux value. In case of failure in the rod control system, the incidence of neutrons becomes uncontrolled and then the volumetric heat source term becomes variable. In this context, the cladding temperature increases and after a certain time the cladding melting occurs and then the nuclear fission products mix with the coolant, which can cause a nuclear accident. In addition, in case of failure in the primary heat transport system, radioactive material may be released into the environment and cause irreversible damage to human life.

The fuel rod melting behavior during a nuclear accident is basically determined by solving the heat conduction equation. Therefore, if the time and spatial variation of the heat source is known, the critical time - defined as the period required for the cladding melting temperature to be reached - can be evaluated. Then, an accurate description of the transient temperature distribution in the fuel rod is essential for predicting the thermal behavior in a nuclear reactor in the event of a nuclear accident.

This paper addresses an analytical analysis for the transient heat transfer with variable heat source in nuclear fuel rods by employing the Classical Integral Transform Technique, and the developed solution can suit any type of variation of the heat source. This solution is compared with the one obtained by Bhattacharya *et al.* (2001), in which Green's functions were used for solve the heat conduction problem. Aspects of the convergence rate of the developed solution together with the influence of the Biot number in the critical time are also reported.

## 2. MATHEMATICAL FORMULATION

In order to determine the transient temperature distribution in the nuclear fuel rod problem, we start by considering the general heat diffusion equation subjected to the commonly accepted simplifying hypothesis: radial one dimensional situation together with a constant source term. Therefore, the partial differential equation for the temperature field is:

$$\rho c_p \frac{\partial T}{\partial t'} = \frac{k}{r'}\frac{\partial}{\partial r'}\left(r'\frac{\partial T}{\partial r'}\right) + G \tag{1}$$

The boundary conditions in the radial direction are given by:

$$\frac{\partial T(t',0)}{\partial r'} = 0 \tag{2}$$

$$k\frac{\partial T(t',R)}{\partial r'} + h(T - T_c) = 0 \tag{3}$$

Implicit in the two relations above is the fact that the fuel rod is taken as a solid element, which is exchanging heat to an environment by a convective process. Also apparent in Eq. (3) is the consideration that both the air gap between the solid element and the clad together with the heat conduction in the clad itself is taken into account in the convective heat transfer coefficient as a first approximation.

Our next step is to express the problem in dimensionless form. This is readily done by utilizing the following parameters:

$$\theta = \frac{T - T_c}{T_c} \tag{4}$$

$$r = \frac{r'}{R} \tag{5}$$

$$t = \frac{kt'}{\rho c_p R^2} \tag{6}$$

$$G^* = \frac{GR^2}{kT_c} \tag{7}$$

Therefore, the problem to be addressed in this contribution takes the form:

$$\frac{\partial \theta}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) + G^* \tag{8}$$

$$\frac{\partial \theta(t,0)}{\partial r} = 0 \tag{9}$$

$$\frac{\partial \theta(t,1)}{\partial r} + Bi\,\theta = 0 \tag{10}$$

The initial condition for the above problem is evaluated by taking into account that a steady state situation is reached under normal, well-controlled operation which requires the source term to be uniform. It is a simple matter to show that:

$$\theta = \frac{G^*}{4}\left(1 - r^2\right) + \frac{G^*}{2Bi} \tag{11}$$

For the case of uncontrolled neutron flux, which is the main concern of this contribution, the volumetric heat source term becomes non-uniform and for a general situation, it can have both space and time dependence. Therefore, the governing equation is given by:

$$\frac{\partial \theta}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) + G'(r,t) \tag{12}$$

Consequently, Eq. (12) together with boundary conditions, Eqs. (9 and 10) and subjected to the initial temperature distribution, Eq. (11), define the problem under scrutiny. In this paper, a set of four different source terms are investigated. The governing equation given by Eq. (12) is non-homogeneous, while that for the boundary conditions are homogeneous. Due to the non-homogeneity of this thermal problem, the solution of partial-differential equation of heat conduction by the classical method of separation of variables is impracticable. Then, the problem is solved analytically by the Classical Integral Transform Technique.

According to Ozisik (1993), this method is based on an eigenfunction expansion in which the auxiliary problem is the regular Sturm-Liouville eigenvalue problem for the heat diffusion in cylindrical coordinates. The main goal of this procedure is to transform the original problem in a set of uncoupled ordinary differential equations, which can be solved in a fully analytical fashion due to the particular nature of the problem here analyzed. At this point, an inverse relation is called in order to determine the original dimensionless temperature distribution.

Therefore, in this method is desired to express the solution $\theta(r,t)$ in terms of an expansion of the eigenfunctions related to the homogeneous problem of eigenvalue in the considered domain:

$$\theta(r,t) = \sum_{m=1}^{\infty} A_m(t) R_0\left(\beta_m, r\right) \tag{13}$$

We apply the orthogonality property of the eigenfunctions $R_0(\beta_m,r)$ in order to obtain a general expression for the coefficients $A_m(t)$:

$$A_m(t) = \frac{1}{N(\beta_m)} \int_0^1 r\theta(r,t) R_0\left(\beta_m, r\right) dr \tag{14}$$

Then, Eq. (13) is rewritten as:

$$\theta(r,t) = \sum_{m=1}^{\infty} \frac{R_0\left(\beta_m, r\right)}{N(\beta_m)} \int_0^1 r\theta(r,t) R_0\left(\beta_m, r\right) dr \tag{15}$$

The pair of integral transform with respect to the variable $r$ for the function $\theta(r,t)$, is given by:

$$\overline{\theta_m(t)} = \int_0^1 r\theta(r,t) R_0\left(\beta_m, r\right) dr \quad \longrightarrow \quad \text{“Transform”} \tag{16}$$

$$\theta(r,t) = \sum_{m=1}^{\infty} \frac{R_0\left(\beta_m, r\right)}{N(\beta_m)} \overline{\theta_m(t)} \quad \longrightarrow \quad \text{“Inverse”} \tag{17}$$

After obtaining the integral transform pair, we transform the partial differential equation, Eq. (12), into a set of ordinary differential equations, according to Ozisik (1993). Then, the resulting set of ordinary differential equation is given by:

$$\frac{d\overline{\theta_m(t)}}{dt} + {\beta_m}^2 \overline{\theta_m(t)} = \int_0^1 rG'(r,t)R_0(\beta_m, r)dr = \overline{G_m(t)} \tag{18}$$

Moreover, the transformed initial condition is:

$$\overline{\theta_m(0)} = \int_0^1 r\left(\frac{G^*}{4}(1-r^2) + \frac{G^*}{2Bi}\right)R_0(\beta_m, r)dr = \overline{f_m} \tag{19}$$

The formal solution for the ordinary differential equations system, seen in Eqs. (18 and 19), is given by:

$$\overline{\theta_m(t)} = e^{-\beta_m^2 t}\left[\overline{f_m} + \int_0^t \overline{G_m(t)} e^{\beta_m^2 \tau} d\tau\right] \tag{20}$$

By applying the inverse of the transform, Eq. (17), we obtain the solution for the dimensionless temperature profile, given by:

$$\theta(r,t) = \sum_{m=1}^{\infty} \frac{R_0(\beta_m, r)}{N(\beta_m)} e^{-\beta_m^2 t}\left[\int_0^1 r\left(\frac{G^*}{4}(1-r^2) + \frac{G^*}{2Bi}\right)R_0(\beta_m, r)dr + \int_0^t \int_0^1 rG'(r,t)R_0(\beta_m, r)e^{\beta_m^2 \tau} dr d\tau\right] \tag{21}$$

The positive roots of following transcendental equation give the eigenvalues βm:

$$\beta_m J_1(\beta_m) = Bi J_0(\beta_m) \tag{22}$$

And $R_0(\beta_m)$ e $1/N(\beta_m)$ are respectively the eigenfunctions and the inverse of the norm for the eigenvalue problem, given by:

$$R_0(\beta_m, r) = J_0(\beta_m r) \tag{23}$$

$$\frac{1}{N(\beta_m)} = \frac{2}{J_0^2(\beta_m)} \frac{{\beta_m}^2}{({\beta_m}^2 + Bi^2)} \tag{24}$$

By introducing Eqs. (23 and 24) into Eq. (21) and solving for the first integral on the right hand side of the equation, the solution for the problem can be rewritten as:

$$\begin{aligned} \theta(r,t) &= 2G^* \sum_{m=1}^{\infty} \frac{Bi}{{\beta_m}^2({\beta_m}^2 + Bi^2)} \frac{J_0(\beta_m r)}{J_0(\beta_m)} e^{-\beta_m^2 t} \\ &+ 2\sum_{m=1}^{\infty} \frac{{\beta_m}^2}{({\beta_m}^2 + Bi^2)} \frac{J_0(\beta_m r)}{{J_0}^2(\beta_m)} e^{-\beta_m^2 t} \int_{\tau=0}^{\tau=t} \int_{r=0}^{r=1} rG'(r,t) J_0(\beta_m r) e^{\beta_m^2 \tau} dr d\tau \end{aligned} \tag{25}$$

Therefore it can be observed that for any kind of variation of heat source with respect to time and space, $G'(r,t)$, the integral appearing in the solution of the temperature distribution, Eq. (25), can be evaluated. For the case of a more involved variation, this integral can be evaluated numerically and ultimately the temperature distribution is obtained. Bhattacharya *et al.* (2001) proposed two kinds of variation for the heat source term, as seen in cases 1 and 2, and from these, the temperature profile were evaluated.

Case 1: $G' = G^* t$

$$\theta(r,t) = 2G^*\sum_{m=1}^{\infty}\frac{Bi}{\beta_m{}^2(\beta_m{}^2+Bi^2)}\frac{J_0(\beta_m r)}{J_0(\beta_m)}e^{-\beta_m^2 t} + 2G^*\sum_{m=1}^{\infty}\frac{Bi}{\beta_m{}^2(\beta_m{}^2+Bi^2)}\frac{J_0(\beta_m r)}{J_0(\beta_m)}\left(t-\left(\frac{1-e^{-\beta_m^2 t}}{\beta_m{}^2}\right)\right) \tag{26}$$

Case 2: $G' = G^* r^2 e^{c_3 t}$

$$\theta(r,t) = 2G^*\sum_{m=1}^{\infty}\frac{Bi}{\beta_m{}^2(\beta_m{}^2+Bi^2)}\frac{J_0(\beta_m r)}{J_0(\beta_m)}e^{-\beta_m^2 t} + 2G^*\sum_{m=1}^{\infty}\frac{J_0(\beta_m r)}{J_0(\beta_m)}\left(\frac{e^{c_3 t}-e^{-\beta_m^2 t}}{c_3+\beta_m{}^2}\right)\left(\frac{Bi+2-\dfrac{4Bi}{\beta_m{}^2}}{\beta_m{}^2+Bi^2}\right) \tag{27}$$

For the cases in analysis, the solutions obtained analytically by the Classical Integral Transform Technique, Eqs (26 and 27), were the same obtained by Bhattacharya *et al.* (2001) using Green's functions, which shows the great agreement between both methods. We observe that the initial condition, Eq. (11), depends only on the spatial coordinate, *r*, the constant heat source term, $G^*$, which it is known, and on the Biot number. Then, for any kind of variable heat source proposed, the initial condition is the same. In this work, two new variations for the heat source *G'* are proposed, given by cases 3 and 4.

Caso 3: $G' = G^*(1+c_1 t)$

$$\theta(r,t) = 2G^*\sum_{m=1}^{\infty}\frac{Bi}{\beta_m{}^2(\beta_m{}^2+Bi^2)}\frac{J_0(\beta_m r)}{J_0(\beta_m)}e^{-\beta_m^2 t} + 2G^*\sum_{m=1}^{\infty}\frac{Bi}{\beta_m{}^2(\beta_m{}^2+Bi^2)}\frac{J_0(\beta_m r)}{J_0(\beta_m)}\left(c_1\left(t-\frac{1}{\beta_m{}^2}\right)+e^{-\beta_m^2 t}\left(\frac{c_1}{\beta_m{}^2}-1\right)+1\right) \tag{28}$$

Caso 4: $G' = G^*(1+c_2 r^2)e^{c_3 t}$

$$\theta(r,t) = 2G^*\sum_{m=1}^{\infty}\frac{Bi}{\beta_m{}^2(\beta_m{}^2+Bi^2)}\frac{J_0(\beta_m r)}{J_0(\beta_m)}e^{-\beta_m^2 t} + 2G^*\sum_{m=1}^{\infty}\frac{J_0(\beta_m r)}{J_0(\beta_m)}\left(\frac{e^{c_3 t}-e^{-\beta_m^2 t}}{c_3+\beta_m{}^2}\right)\left(\frac{Bi+c_2\left(Bi+2-\dfrac{4Bi}{\beta_m{}^2}\right)}{\beta_m{}^2+Bi^2}\right) \tag{29}$$

## 3. RESULTS AND DISCUSSION

According to El- Wakil (1962), in a typical uranium nuclear fuel rod the following data can be considered in the thermal analysis: average thermal conductivity of uranium ($k$=31.33 W/mK$^{-1}$); average density ($\rho$=18.76x10$^3$ kg/m³); average specific heat ($C_p$=136.4 J/kgK$^{-1}$); fuel rod radius ($R$=0.03 m); uniform volumetric heat source ($G$=2.256x10$^8$ W/m³ ); coolant temperature ($T_c$=200ºC); melting point of the cladding material (stainless steel-304) is taken as 1426 ºC. From this data, the dimensionless uniform heat source, $G^*$, is equal to 32.4.

The analytical solution obtained here by the Classical Integral Transform Technique was the same obtained by Bhattacharya *et al.* (2001) using Green's functions. The validation of analytical solution was presented in the work of Bhattacharya *et al.* (2001) through a comparison to a numerical solution, and thus it shall not be repeated in this work.

Figures 1.a and 1.b depict the dimensionless transient temperature profile for cases 1 and 3, respectively. The temperature profile was plotted for $Bi$=15 in four different times. We observe that for a specific time, the temperature at the center of the rod is a maximum and decreases towards the clad surface, as expected, due to the convective cooling in the element surface ($r$=1).

However, a close inspection of Fig. 1.a shows a physical incoherence associated to the thermal problem in the nuclear reactor. After the initial condition ($t$=0) is expected that for the same radial position, the temperature rises increasingly due to uncontrolled neutron flux, which increases the heat generation of the fuel element. However, as the variable heat source term treated in case 1, it only reaches the value of $G^*$ (referring to the uniform heat source in steady state) at $t$ equal to 1. Consequently, for $t$<1, the variable heat source term is smaller than the uniform heat source term, and then the local temperature is lower than the initial temperature, which is contradictory with the physics of the problem. This fact is evident when analyzing the curves shown in the Fig. 1.a, where it is observed that at $t$=0,5 and $t$=1,0, the local temperature is lower than the initial temperature of the fuel element, which it is exceeded only at $t$=1,5.

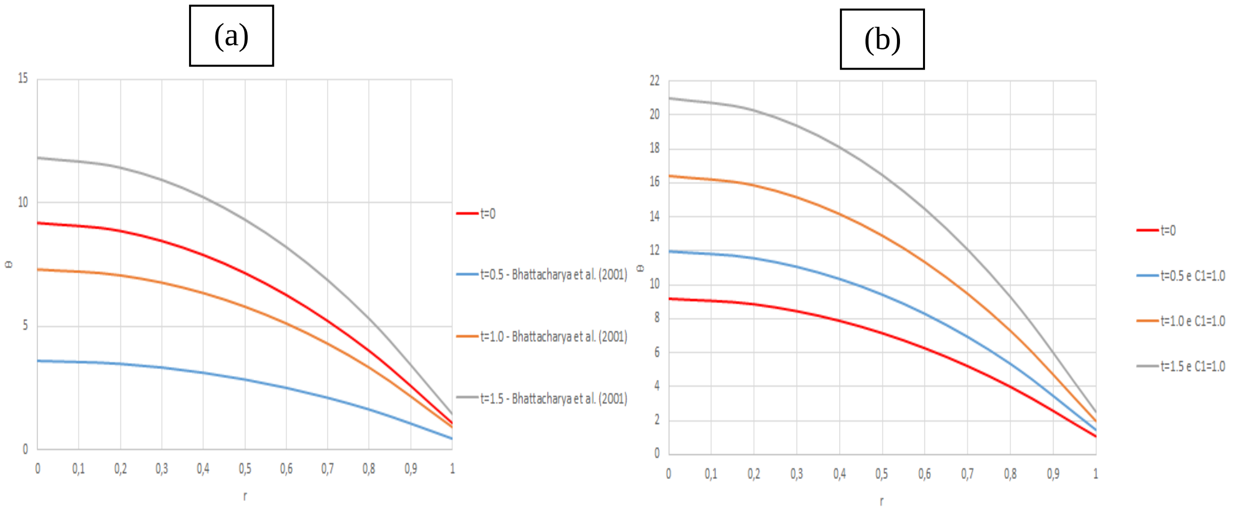


Figure 1. Temperature profiles for $Bi$=15 (a) Case 1. (b) Case 3 for $c_1$=1.

In order to obtain a better physical representation of the thermal problem in the fuel rod, this work proposes another variation for the heat source with respect to time, expressed in case 3. In this analysis, the transient solution starts from the steady state with uniform heat source across the fuel element, and as time increases, the heat source increases from $G^*$, as expected. Fig. 1.b is similar to Fig. 1.a, but for the variation of heat source term proposed in case 3. By analyzing Fig. 1.b, we note that as time increases the temperature in the same radial position rises, and clearly the closer to the cylinder surface, the smaller is the temperature variation for two consecutive time instants. On the other hand, opposed to what was observed in case 1, for case 3 proposed in this work, the temperature profile starts from the initial condition (t=0) and rises continuously as time increases, reproducing better the physics of the thermal problem of a nuclear reactor. This fact is evident when we analyze the curves shown in Fig. 1.b.

Another point worth mentioning is in regard of constant $c_1$, whose role becomes clear. The higher the value of this constant, the higher local temperatures are observed in the fuel rod, as expected, since when $c_1$ increases, the variable heat source term expressed by case 3 increases.

Figure 2 reports the influence of the Biot number with respect to the temperature profile in the fuel rod. We note that for a specific time, the local temperature is smaller for the case of higher Biot numbers. Clearly, this behavior is expected, since the higher the Biot number, the larger is the convection heat transfer in the fuel element surface.

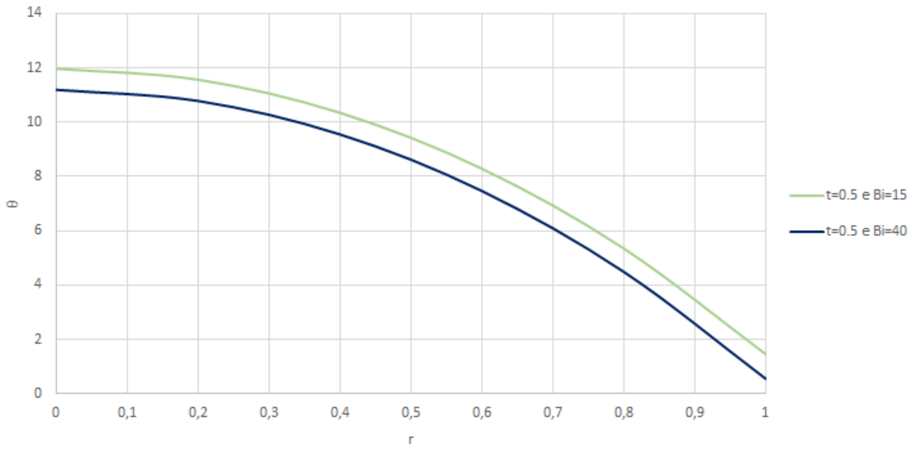


Figure 2. Temperature profile for case 3 with $c_1$=1 and $t$=0,5, for $Bi$=15 and $Bi$=40.

The variable heat source term proposed in case 2 by Bhattacharya *et al.* (2001) is more involved, since it is not uniform anymore in the radial direction. In this approach, the heat source increases along the radial direction and therefore, the cylinder surface is the radial position with higher heat generation. Consequently, closer to the cylinder center, the heat source is smaller than the uniform heat source, $G^*$, in the event of $r^2e^{\ c3t}$ be less than 1. Evidently, for small times this situation occurs, and then, as already observed case 1 for small times, again the local temperature is smaller than the initial temperature in the fuel rod, which is contradictory with the physical of the thermal problem. This fact is evident when we analyze the curves shown in Fig. 3.a, where it is observed that for $t$=0,05 and $t$=0,25, the local temperature is smaller than initial temperature of the fuel rod, mainly for shorter values of $r$, where the heat generation is smaller.

Obviously, as time increases this fact is overcome and an interesting aspect is observed, which is that despite the heat generation presents it largest value near the element surface, the maximum temperature is attained in the intermediate value of $r$, as can be observed in Fig. 3.a at $t$=0,50. This fact should not be a surprise since a convective cooling occurs at the cylinder surface, and as the conduction resistance is bigger than the convection resistance due to Biot number be greater than 1 ($Bi$=15), the energy transport by conduction is not fast enough in the radial direction, generating an energy accumulation in the intermediate region, where the maximum temperature is observed.

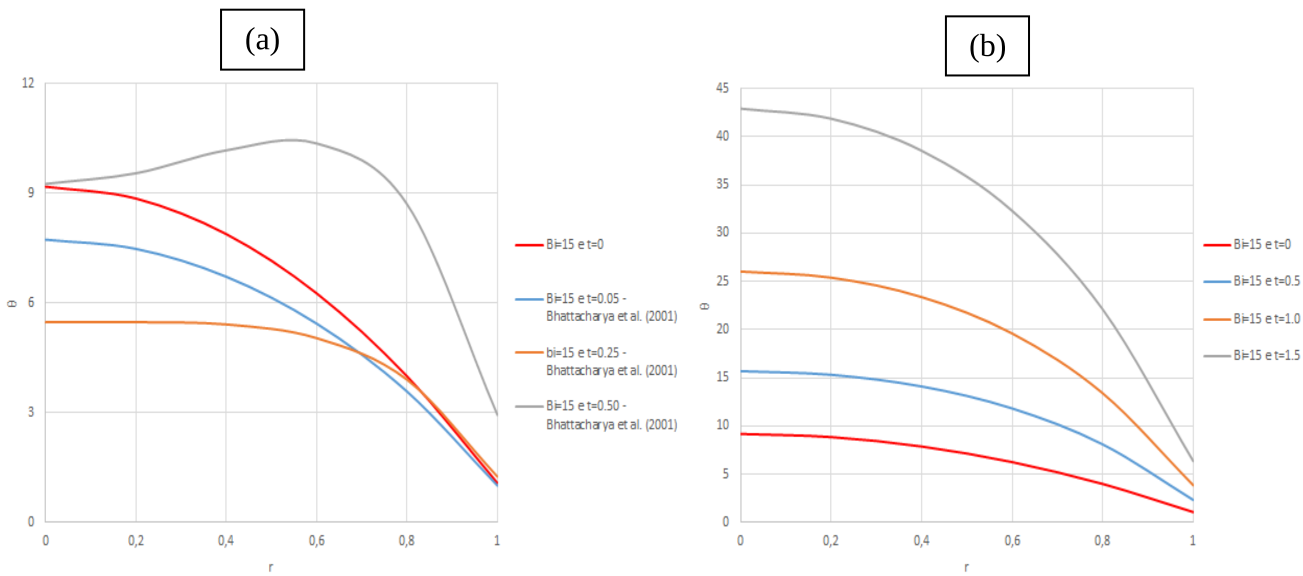


Figure 3. Temperature profiles for $Bi$=15 (a) Case 2 for $c_3$=4. (b) Case 4 for $c_2$=1 and $c_3$=1.

In order to obtain a better physical representation of the thermal problem in the fuel rod, this work proposes another variation for the heat source with respect to time and space, expressed in case 4. In this case, the transient solution starts from steady state and even for $r$=0 (cylinder center), and as time increases, the variable heat source increases from $G^*$, as expected in the thermal problem under scrutiny. Fig. 3.b depicts the local temperature profile for the variation of the heat source proposed case 4. By analyzing each curve on this graph, we can see that as time increases, the local temperature increases continuously, which was not observed in the analysis for case 2.

Another feature deals with the behavior of $c_2$, where it is observed that by increasing widely its value, a bigger variation of the heat source in the radial direction is obtained, and therefore, the situation presented in Fig. 3.a is reproduced, since as time increases, the highest temperatures are found in the intermediate region of the fuel rod. The role of $c_3$ is also clear, since as its value increases, higher local temperatures are found.

A parameter of great engineering interest is the temperature profile in the cladding material, which is the temperature distribution along the time at r=1. From this, the critical time, which is the period required for the cladding to reach its melting temperature, can be evaluated. Figure 4 shows the critical time in relation to the Biot number for the two variations of heat source proposed in this work.

The analysis of this graphic presents the influence of Biot number in relation to the local temperature profile in the fuel rod. The higher the Biot number, the lower the temperature along the cylinder and consequently, a longer time period will be required for the cladding to reach its melting temperature. Therefore, as the Biot number increases, the higher is the critical time. Such behavior was observed for all the situations explored in Fig. 4. It is observed that for case 4, the increase in the Biot number does not ensure a significant decrease of the critical time, and as a result, another way of intensifying the cooling in the cylinder surface could be evaluated.

Another aspect observed in Fig. 4 is that for both cases analyzed, cases 3 and 4, as the value of the constants present in the functions of variable heat source increases, the critical time decreases, that is, the cladding melts faster. This fact is expected, since by increasing the value of the constants, the heat source term raises and consequently, the local temperature in the fuel element increases. In terms of dimensional time, for case 3 with $C_1$=0.5, the critical time is equal to 7.11 minutes for Bi=10 and 34.79 minutes for Bi=40. Therefore, the influence on the Biot number in the analysis of the thermal problem is clear. Besides, Fig. 4 suggests that an accurate modeling of the variable heat source is mandatory, since it shows that cases 3 and 4 may lead to very different critical times, especially as the Biot number increases.

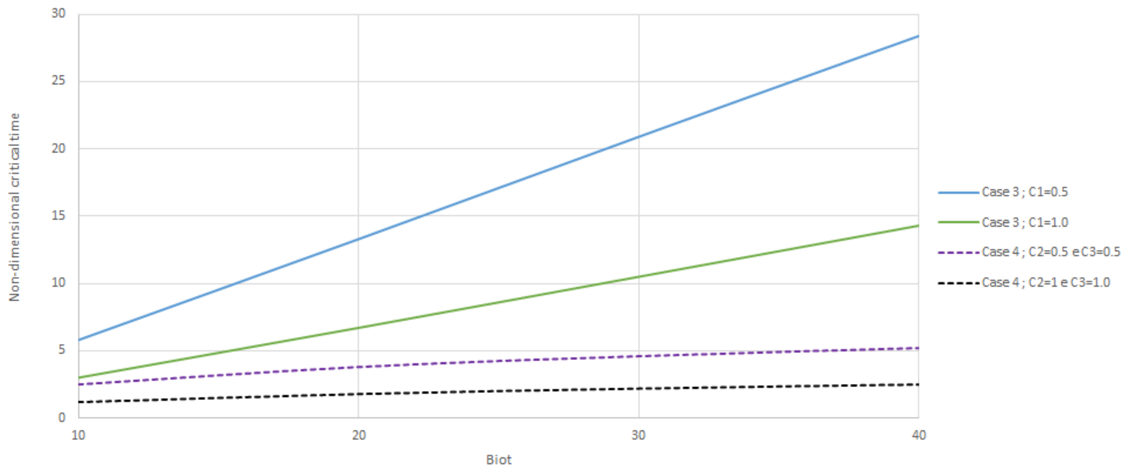


Figure 4. Variation of critical time in relation to the Biot number for cases 3 and 4.

Another interesting approach for the temperature profile in the cladding material is to check the convergence of this solution for the cases proposed here. For both cases proposed, cases 3 and 4, the solution for the local temperature is divided in two sums, as seen in Eq. (28) and Eq. (29), and we observe that the first sum on the right hand side of each of these equations are the same for both cases. Clearly, such fact is expected, since this first sum refers to the solution of the homogeneous problem, which does not contain the heat source. The second sum seen in these equations refers to the solution of the non-homogeneous problem with heat source, and it is specific for each case analyzed, and therefore, should present different convergence aspects.

An accuracy of six significant digits in the convergence analysis is sought, in other words, when the difference between two consecutive terms of the sum is smaller than $10^{-6}$, the result is considered to be converged. Table 1 reports the convergence of the sum for the homogeneous problem, which is the same for both cases discussed. This solution is evaluated at r=1, with *Bi*=15.

Table 1. Convergence analysis for the sum of the homogeneous problem at *r*=1 and *Bi*=15.

| Dimensionless time (t) | $1x10^{-6}$ | $1x10^{-5}$ | $1x10^{-4}$ | $1x10^{-3}$ | $1x10^{-2}$ | $1x10^{-1}$ | 1 |
|---|---|---|---|---|---|---|---|
| Number of terms in the sum | 11 | 11 | 10 | 9 | 6 | 3 | 2 |

It can be noticed that this sum has an eigenvalue squared under a negative exponential with respect to time. Since the eigenvalues are all positive and form a growing sequence, they tend to lead this sum to converge. Due to the negative exponential, as time increases, the trend is that the sum shows a faster convergence. Such fact can be observed on Tab. 1, where as time increases, the amount of terms needed to reach the desired accuracy decreases, and from $t$=1, the sum already shows converged results with only one term. In general, the sum appearing in the homogeneous problem presents a fast convergence even for low values of $t$, much due to the eigenvalues behavior mentioned previously.

Table 2 depicts the convergence analysis of the sum appearing on the non-homogeneous problem case 3, which is the second sum on the right hand side of Eq. (28), evaluated for $c_1$=1. We note that now, the amount of sum terms needed to reach the desired accuracy increases as time increases. Such fact is expected, since the variable heat source is growing with respect to time. Obviously, $t$=100 is much higher than the critical time, which would be of interest to evaluate. In general, the analyzed sum shows a fast convergence behavior even as time increases greatly.

Table 2. Convergence analysis for the sum of the non-homogeneous problem for case 3 at $r$=1 and $Bi$=15.

| Dimensionless time (t) | $1x10^{-6}$ | $1x10^{-5}$ | $1x10^{-4}$ | $1x10^{-3}$ | $1x10^{-2}$ | $1x10^{-1}$ | 1 | 10 | 100 |
|---|---|---|---|---|---|---|---|---|---|
| Number of terms in the sum | 2 | 2 | 2 | 9 | 11 | 11 | 13 | 19 | 33 |

Table 3 reports the convergence analysis of the sum appearing on the non-homogeneous problem for case 4, which is the second sum on the right hand side of Eq. (29), evaluated for $c_2$=1 and $c_3$=1. As already discussed in the analysis of case 3, the amount of terms in the sum needed to reach the desired accuracy increases as time increases, but here at a higher rate when compared to the previous case. Clearly, from Fig. 4 it is possible to verify that the critical time for the analysis addressed on Tab. 3 is approximately $t$=1.4, and therefore, for this time, the convergence of the sum is obtained with few terms. Obviously, as time largely increases, a higher amount of terms is needed to obtain converged results within the established accuracy.

Table 3. Convergence analysis for the sum of the non-homogeneous problem for case 4 at $r$=1 and $Bi$=15.

| Dimensionless time (t) | $1x10^{-6}$ | $1x10^{-5}$ | $1x10^{-4}$ | $1x10^{-3}$ | $1x10^{-2}$ | $1x10^{-1}$ | 1 | 10 | 100 |
|---|---|---|---|---|---|---|---|---|---|
| Number of terms in the sum | 2 | 4 | 17 | 25 | 25 | 26 | 32 | 293 | 21859 |

## 4. CONCLUSIONS

In this work an analytical solution was developed through the Classical Integral Transform Technique for the temperature distribution in a nuclear fuel rod with variable heat source term. It is observed that the Integral Transform Technique and Green's functions have good agreement in the solution of thermal problem, since the same analytical solution was found with both methods.

The analytical expression developed by Integral Transform Technique in this article is extremely useful for thermal analysis in nuclear fuel rods, because regardless of the form of the variable heat source (with respect to time and space) is treated, the temperature profile can be obtained. In addition, the solution developed in this paper presents great convergence aspects, since the rate of convergence for the eigenfunctions expansions are found to be very fast and for the cases analyzed here, a 40 term expansion is able to predict fully converged results for a six digit approximation. Therefore, this tool is believed to be a good alternative to more involved approaches such as purely numerical methods based on the finite element or finite difference approximation.

From a practical point of view of engineering, the critical time, which is an important parameter in the thermal analysis of the nuclear reactor, can be evaluated through the surface temperature profile. In this context, Biot number can play an important role, as increasing the Biot number raises the critical time, and therefore the cladding material takes longer for melting, which can be important in a nuclear accident.

## 5. ACKNOWLEDGEMENTS

The authors are thankful to IME and CAPES for the all support received during the development of this work.

## 6. REFERENCES

Bhattacharya, S.; Nandi, S.; DasGupta, S; Sirshendu, D. 2001. Analytical Solution of Transient Heat Transfer with Variable Source for Applications in Nuclear Reactors. Int. Communications in Heat and Mass Transfer, N. 7, pp. 1005-1013, 2001.

An, C.; Moreira, F.C.; Su, J. 2014. Thermal analysis of the melting process in a nuclear fuel rod. Apllied Thermal Engineering 68 (2014) 133-143.

El--Wakil, M.M. Nuclear Power Engineering, McGraw-Hill, New York, 1962.

Eskandari, M.R; Bavandi, A.; Mihandoost, A., 2012. Studies on Nuclear Fuel Rod Thermal Performance. Energy Procedia 14 (2012) 142-147.

Ozisik, M.N. 1993. Heat Conduction. 2nd ed., Wiley, New York, 716 p.

## 7. NOMENCLATURE

| | |
|---|---|
| G | Volumetric Heat source, W/m³ |
| $G^*$ | Non-dimensional uniform heat source |
| G' | Non-dimensional variable heat source |
| h | Heat transfer coefficient, $W/m^2K^{-1}$ |
| k | Thermal conductivity, $W/mK^{-1}$ |
| ρ | Density, $Kg/m^3$ |
| $C_p$ | Specific heat, $J/kgK^{-1}$ |
| T | Temperature, ºC |
| t' | Temporal coordinate, s |
| r' | Spatial coordinate , m |
| θ | Non-dimensional Temperature |
| $T_c$ | Coolant temperature, ºC |
| R | Radius of fuel rod, m |
| Bi | Biot number |
| r | Non-dimensional spatial coordinate |
| t | Non-dimensional temporal coordinate |
| $\beta_m$ | Eigenvalues |
| $R_0(\beta_m)$ | Eigenfunctions |
| $N(\beta_m)$ | Norm |
| $A_m(t)$ | Temporal coefficients |
| $\overline{\theta_m(t)}$ | Non-dimensional transformed temperature |
| $\overline{G_m(t)}$ | Transformed heat source |
| $\overline{f_m(t)}$ | Transformed initial condition |

## 8. RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.