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DYNAMIC RESPONSE OF TIMOSHENKO BEAMS ON PASTERNAK FOUNDATION

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Abstract. *The dynamic response of beams on elastic foundation has been extensively studied in the modern engineering, especially in the fields of transportation systems. This approach is mainly used to model a railroad track. In this paper, a finite element is developed using cubic and quadratic polynomials. The finite element and the analytic solutions are compared to verify the reliability of the method in this kind of problem. The influence of the foundation parameters and the boundary conditions configuration are taken into account. The results showed that Pasternak foundation increases the frequency parameters of the beam. Also, the presence of the foundation reduces the normal mode amplitude and the clamped-free case is the most affected by the presence of a Pasternak foundation.*

Keywords: *Timoshenko Beam Theory, Finite Element Method, Elastic Foundation, Free Vibration Analysis, Pasternak Foundation Model.*

1. INTRODUCTION

Beams on elastic foundation is a continually discussed subject in modern engineering as it constitutes a practical idealization of many problems, such the railway track design and maintenance. However, for an efficient modelling, it is fundamental to consider the mechanical behavior of the beam, the behavior of soil media and the interaction between them. Thus, to consider these subsystems of distinct nature results in a great mathematical complexity in the modeling of structure-foundation interactions.

The onerous process of understanding all the physical behavior of a soil media, due to the intrinsic complexity of the soil response, lead to the development of many idealized models, which reduces the analytic rigor expended in the problem and provides reasonable results (Avramidis and Morfidis, 2006).

The mechanical approach model the soil response with purely elastic characteristics and the resulting displacement is given by linear functions. The first mechanical model was proposed by Winkler (1867). This model considers the soil as a system of mutually independent spring elements in which the displacement is proportional to the loaded region. However, Winkler foundation disregards the effect of interaction between the springs, hence, the soil response presents no cohesion.

To confront this limitation, researchers suggested various models which regard the continuity of the soil media, such Hetenyi (1946), Filonenko-Borodich (1940), Pasternak (1954) and Kerr (1965). Kerr (1964) showed that the Pasternak foundation model is the most natural extension of the Winkler model for homogeneous foundations, in which the soil cohesion is given by a shear layer of incompressible vertical elements that resist only to transverse shear that is attached to the end of the springs (Pasternak, 1954).

In many engineering applications, the mechanical behavior of the beam bases in the Euler-Bernoulli beam theory, in which straight lines and normal planes remain straight and normal after the deformation. This theory neglects the effect of shear deformations and rotatory inertia, which influence can be well delineated for higher modes and thicker beams, hence the Euler-Bernoulli beam theory can be applied with good approximation only for slender beams and lower frequencies (Soares and Hoefel, 2015). To overcome this limitation, Timoshenko (1921) proposed a model which considers the effect of transverse shear deformation and rotatory inertia that provides an excellent approximation of the beam behavior.

Over the years, researches have concentrated their efforts to better understand the soil-structure interaction. Wang and Stephens (1977) studied the natural vibrations of finite Timoshenko beams on Pasternak foundations and derived the frequency equations for some boundary conditions. Wang and Gagnon (1978) presented the effects of shear deformation, rotatory inertia, and the foundation factor on the dynamic response of continuous Timoshenko beams on Winkler and

Pasternak foundations. Yokoyama (1996) presented a finite element for the free vibration of a uniform Timoshenko beam column on a two-parameter elastic foundation. Thambiratnam and Zhuge (1996) develops a simple finite element method and applies in the treatment of the free vibration of stepped beam and continuous beam on an elastic foundation, and beam on a stepped elastic foundation. De Rosa (1995) examined the free vibration frequencies of Timoshenko beams on two-parameter elastic foundation. The fundamental natural frequencies of vibration of finite Timoshenko beams on Pasternak foundation by Rayleigh's principle was derived by El-Mously (1999). Ghannadiasl and Mofid (2015) presented the exact solution to free vibration of elastically restrained Timoshenko beam on an arbitrary variable elastic foundation using Green functions.

The purpose of this paper is to analyze the influence of Pasternak foundation in the free vibration response of a Timoshenko beam. A two-node beam element with two degree of freedom per node with cubic and quadratic polynomials for transverse displacement and slope due bending, respectively, is developed. The effect of the boundary conditions configurations and the foundation parameters are investigated.

2. CLASSICAL THEORY

Figure 1 shows a scheme of a uniform Timoshenko beam on a Pasternak foundation. By using Hamilton principle and employing the Timoshenko beam theory, one can obtain two coupled differential equations for free vibration response (Wang and Stephens, 1977) :

$$\rho A \frac{\partial^2 \nu(x,t)}{\partial t^2} + \kappa A G \left(\frac{\partial \psi(x,t)}{\partial x} - \frac{\partial^2 \nu(x,t)}{\partial x^2} \right) + k_f \nu(x,t) - G_p \frac{\partial^2 \nu(x,t)}{\partial x^2} = 0, \tag{1}$$

$$EI \frac{\partial^2 \psi(x,t)}{\partial x^2} - \rho I \frac{\partial^2 \psi(x,t)}{\partial t^2} - \kappa A G \left(\psi(x,t) - \frac{\partial \nu(x,t)}{\partial x} \right) = 0. \tag{2}$$

where A is the cross-sectional area, I, the moment of inertia of cross section, ρ is the mass per unit volume, E, the modulus of elasticity, G the modulus of rigidity, κ is the shape factor or shear coefficient, k_f the foundation elastic stiffness coefficient, G_p the foundation shear layer stiffness,and $\nu(x,t)$ and $\psi(x,t)$ are the transverse deflection and the beam slope due to bending, respectively, at the axial location x and time t.

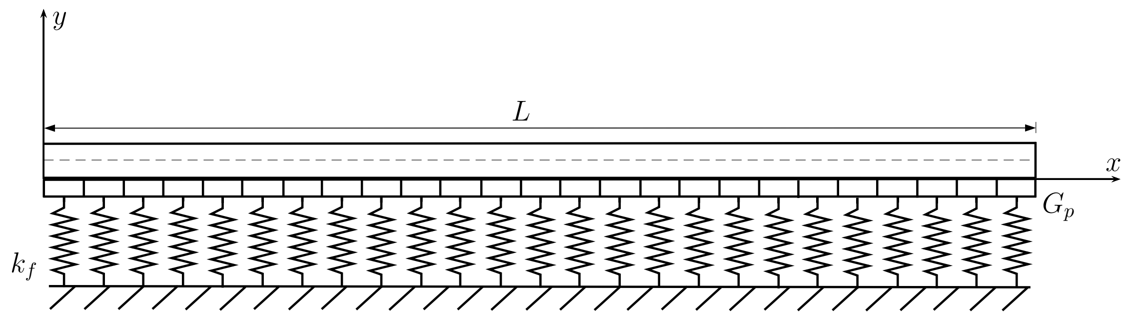


Figure 1. A beam on a Pasternak foundation.

Differentiating Eq. (2) and substituting in Eq. (1), and assuming that the beam is excited harmonically, in which:

$$\nu(x,t) = V(x)e^{i\omega t}, \quad \psi(x,t) = \Psi(x)e^{i\omega t},$$

$$\xi = x/L, \quad b^2 = \frac{\rho A L^4}{EI}\omega^2, \tag{3}$$

where i = $\sqrt{-1}$, L is the length of beam, ξ is the non-dimensional length of the beam and $V(x)$ and $\Psi(x)$ are the normal functions of $\nu(x,t)$ and $\psi(x,t)$, respectively, one can obtain (Wang and Stephens, 1977; De Rosa, 1995):

$$\frac{d^4 V(\xi)}{d\xi^4} + \gamma \frac{d^2 V(\xi)}{d\xi^2} + \zeta V(\xi) = 0, \tag{4}$$

$$\frac{d^4 \Psi(\xi)}{d\xi^4} + \gamma \frac{d^2 \Psi(\xi)}{d\xi^2} + \zeta \Psi(\xi) = 0, \tag{5}$$

where

$$\gamma = \frac{b^2\left(r^2+s^2\right)-s^2e^2+p^2\left(b^2r^2s^2-1\right)}{1+s^2p^2}, \quad \zeta = \frac{\left(b^2-e^2\right)\left(b^2r^2s^2-1\right)}{1+s^2p^2}$$

and r, s, e and p are the coefficients related with the effect of rotary inertia, shear deformation, elastic and shear layer stiffness, respectively, given by:

$$r^2 = \frac{I}{AL^2},\ s^2 = \frac{EI}{\kappa GAL^2},\ e^2 = \frac{k_f L^4}{EI},\ p^2 = \frac{G_p L^2}{EI}. \tag{6}$$

In order to solve the O.D.E. of Eqs. (4) and (5), two conditions must be considered. This conditions represents different solution expressions.

For the first case, $\zeta < 0$, the solutions can be expressed in trigonometric and hyperbolic functions:

$$V(\xi) = C_1\, cosh(\alpha_1\xi) + C_2\, sinh(\alpha_1\xi) + C_3\, cos(\beta\xi) + C_4\, sin(\beta\xi), \tag{7}$$

$$\Psi(\xi) = C_1'\, sinh(\alpha_1\xi) + C_2'\, cosh(\alpha_1\xi) + C_3'\, sin(\beta\xi) + C_4'\, cos(\beta\xi), \tag{8}$$

where:

$$\alpha_1 = \frac{\sqrt{2}}{2}\sqrt{-\gamma+\sqrt{\gamma^2-4\zeta}}, \tag{9}$$

$$\beta = \frac{\sqrt{2}}{2}\sqrt{\ \gamma+\sqrt{\gamma^2-4\zeta}}, \tag{10}$$

and C and C' are constants.

The second case gives $\zeta > 0$. As a result, the solution is expressed only in trigonometric functions:

$$V(\xi) = \bar{C}_1\, cos(\alpha_2\xi) + \bar{C}_2\, sin(\alpha_2\xi) + \bar{C}_3\, cos(\beta\xi) + \bar{C}_4\, sin(\beta\xi), \tag{11}$$

$$\Psi(\xi) = \bar{C}_1'\, sin(\alpha_2\xi) + \bar{C}_2'\, cos(\alpha_2\xi) + \bar{C}_3'\, sin(\beta\xi) + \bar{C}_4'\, cos(\beta\xi), \tag{12}$$

where:

$$\alpha_2 = \frac{\sqrt{2}}{2}\sqrt{\gamma-\sqrt{\gamma^2-4\zeta}}, \tag{13}$$

$$\beta = \frac{\sqrt{2}}{2}\sqrt{\gamma+\sqrt{\gamma^2-4\zeta}}, \tag{14}$$

and $\bar{C}$ and $\bar{C}'$ are constants.

Equations (4) and (5) shows that the beam-foundation theory represents a generalization of the beam theory. Disregarding the parameters e and p, the solution regress to the solution of a beam without foundation. Also, the Pasternak foundation theory is a higher generalization as it includes the solution for Winkler when $p = 0$.

3. FINITE ELEMENT FORMULATION

Consider a uniform Timoshenko beam element on Pasternak Foundation as shown in Fig. 2. The beam element consists of two nodes and each node has two degrees of freedom: V, the total deflection, and Ψ, the slope due to bending.

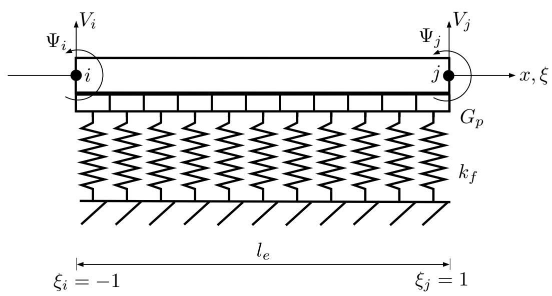


Figure 2. Beam on Pasternak foundation element

Solving the homogeneous form of Timoshenko beam static equations, one can obtain a cubic and quadratic displacement functions as follows (Yokoyama, 1987):

$$V_i(\xi) = \sum_{i=0}^{3}\lambda_i\xi^i \quad \text{and} \quad \Psi_i(\xi) = \sum_{i=0}^{2}\overline{\lambda}_i\xi^i. \tag{15}$$

where λ_i and $\overline{\lambda}_i$ are constants.

Using the non-dimension coordinate, ξ, and element length, l_e, the matrix form of the displacement V and total slope Ψ can be written as:

$$V = [\,\mathbf{N}(\xi)\,]\{\,\mathbf{v}\,\}_e \quad \text{and} \quad \Psi = [\,\overline{\mathbf{N}}(\xi)\,]\{\,\mathbf{v}\,\}_e, \tag{16}$$

where $[\,\mathbf{N}(\xi)\,]$ and $[\,\overline{\mathbf{N}}(\xi)\,]$ are the shape functions and $\{\,\mathbf{v}\,\}_e$ is the vector of nodal coordinates. The subscript e represents expressions for a single element.

Therefore, the shape functions in Eq. (16) can be expressed as:

$$\mathbf{N_i}(\xi) = \frac{1}{4(1+3\beta)} \left\{ \begin{array}{c} 2(3\beta+1) - 3(\beta+1)\xi + \xi^3 \\ (l_e/2)\,[3\beta + 1 - \xi - (3\beta+1)\xi^2 + \xi^3] \\ 2(3\beta+1) + 3(2\beta+1)\xi - \xi^3 \\ (l_e/2)\,[-3\beta - 1 - \xi + (3\beta+1)\xi^2 + \xi^3] \end{array} \right\}, \tag{17}$$

and

$$\overline{\mathbf{N}}_i(\xi) = \frac{1}{4(1+3\beta)} \left\{ \begin{array}{c} (l_e/2)\,(3\xi^2 - 3) \\ -1 - 2(3\beta+1)\xi + 6\beta + 3\xi^2 \\ (l_e/2)\,(3 - 3\xi^2) \\ -1 + 2(3\beta+1)\xi + 6\beta + 3\xi^2 \end{array} \right\}, \tag{18}$$

where $\beta = 4EI/\kappa G A {l_e}^2$.

Thus, considering the foundation and the beam, the potential and kinetic energy for an element length l_e are expressed as:

$$\begin{aligned} \boldsymbol{U_e} \;&=\; \frac{1}{2}\frac{2EI}{l_e}\int_{-1}^{1}\left(\frac{\partial \Psi}{\partial \xi}\right)^2 d\xi + \frac{1}{2}\frac{2\kappa GA}{l_e}\int_{-1}^{1}\left(\frac{2}{l_e}\frac{\partial V}{\partial \xi} - \Psi\right)^2 d\xi \;+ \\ &\quad \frac{1}{2}\frac{k_f l_e}{2}\int_{-1}^{1}(V)^2\, d\xi + \frac{1}{2}\frac{2G_p}{l_e}\int_{-1}^{1}\left(\frac{\partial V}{\partial \xi}\right)^2 d\xi \end{aligned} \tag{19}$$

$$\boldsymbol{T_e} \;=\; \frac{1}{2}\frac{\rho A l_e}{2}\int_{-1}^{1}\left(\frac{\partial V}{\partial t}\right)^2 d\xi + \frac{1}{2}\frac{\rho I l_e}{2}\int_{-1}^{1}\left(\frac{\partial \Psi}{\partial t}\right)^2 d\xi. \tag{20}$$

Substituting the displacement expression, Eq. (16), into the potential energy, Eq. (19), gives:

$$\begin{aligned} \boldsymbol{U_e} \;&=\; \frac{1}{2}\{\mathbf{v}\}_e^T\left[\frac{2EI}{l_e}\int_{-1}^{1}[\overline{\mathbf{N}}(\xi)']^T[\overline{\mathbf{N}}(\xi)']d\xi\right]\{\mathbf{v}\}_e \;+ \\ &\quad \frac{1}{2}\{\mathbf{v}\}_e^T\left[\frac{2\kappa GA}{l_e}\int_{-1}^{1}[\mathbf{N}(\xi)' - \tfrac{l_e}{2}\overline{\mathbf{N}}(\xi)]^T[\mathbf{N}(\xi)' - \tfrac{l_e}{2}\overline{\mathbf{N}}(\xi)]d\xi\right]\{\mathbf{v}\}_e \;+ \\ &\quad \frac{1}{2}\{\mathbf{v}\}_e^T\left[\frac{k_f l_e}{2}\int_{-1}^{1}[\mathbf{N}(\xi)]^T[\mathbf{N}(\xi)]\,d\xi + \frac{2G_p}{l_e}\int_{-1}^{1}[\mathbf{N}(\xi)']^T[\mathbf{N}(\xi)']\,d\xi\right]\{\mathbf{v}\}_e, \end{aligned} \tag{21}$$

where $[\mathbf{N}(\xi)'] = [\partial \mathbf{N}(\xi)/\partial \xi]$.

Substituting the displacement expression, Eq. (16), into the kinetic energy, Eq. (20), gives:

$$\boldsymbol{T_e} = \frac{1}{2}\{\dot{\mathbf{v}}\}_e^T\left[\frac{\rho A l_e}{2}\int_{-1}^{1}[\mathbf{N}(\xi)]^T[\mathbf{N}(\xi)]d\xi + \frac{\rho A l_e}{2}\int_{-1}^{1}[\overline{\mathbf{N}}(\xi)]^T[\overline{\mathbf{N}}(\xi)]d\xi\right]\{\dot{\mathbf{v}}\}_e, \tag{22}$$

4. NUMERICAL RESULTS

In order to study the effect of Pasternak foundations in the natural frequencies of a Timoshenko beam as well as the effect of rotary inertia and shear deformation, a several numerical examples were presented. Thus, consider simply supported beam of uniform cross-section, such that $E = 210\,GPa$, $G = 80.8\,GPa$, $\rho = 7850\,kg/m^3$, $L = 0.5\,m$, $\kappa = 5/6$ and $r = 0.04$.

To analyze the influence of a foundation in the frequency parameters, Tab. 1 presents the comparison between analytic and FEM solutions for a hinged-hinged beam. The second column presents the frequency parameters for a beam without foundation ($e = 0$, $p = 0$), third to sixth, for Winkler foundation ($e = 5$, $p = 0$), and seventh to tenth, for Pasternak foundation ($e = 5$, $p = 5$).

Table 1. Comparison table for the frequency parameters of analytic and FEM analyses.

Mode Number	Without Foundation	Winkler				Pasternak			
		Analytic	FEM-10e	FEM-30e	FEM-70e	Analytic	FEM-10e	FEM-30e	FEM-70e
1	9.5752	10.786	10.787	10.786	10.786	18.962	18.963	18.962	18.962
2	35.410	35.746	35.834	35.756	35.748	47.134	47.206	47.142	47.136
3	71.842	72.003	72.726	72.082	72.018	85.151	85.807	85.222	85.164
4	114.26	114.36	117.22	114.67	114.41	129.18	131.91	129.48	129.24
5	159.85	159.92	167.67	160.78	160.08	176.56	184.11	177.39	176.71

The results showed that the presence of a foundation increase the frequency parameters. Comparing the frequency parameter of a beam on a foundation with the solution for a beam without foundation, the Winkler foundation increase is great for the first mode, but becomes less significant as the mode number rise. However, Pasternak foundation presents a remarkable increase for all mode numbers and higher than those presented by Winkler foundation.

Also, the frequency parameter increase declines as the mode number rise. This reduction can be observed in the frequency parameters of Winkler and Pasternak foundation, however, the decrease in Pasternak is not as high as presented by Winkler.

For higher modes, the difference between the FEM and analytic solutions decreases when the number of elements is increased. Therefore, FEM formulation presents a high accuracy.

The next subsections presents numerical examples for hinged-hinged, clamped-clamped and clamped-free beams for the FEM solution using 70 elements in order to discuss the effect of the Pasternak foundation in the usual boundary condition configurations. The boundary conditions are defined in Tab. 2.

Table 2. Boundary Conditions

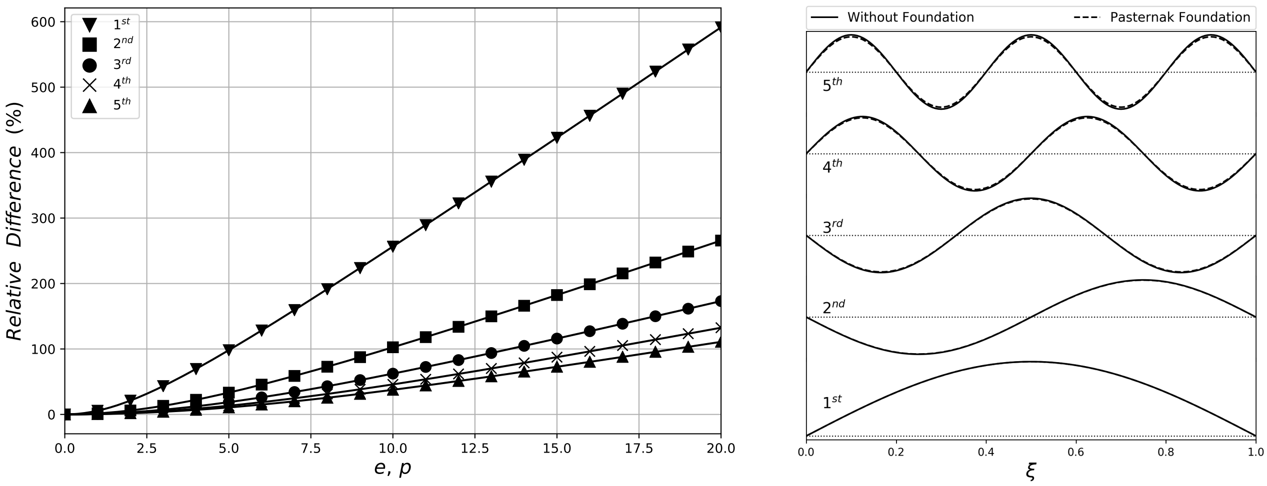
Boundary Condition	Deflection	Slope	Moment	Shear Force
Hinged	$V(\xi) = 0$	-	$\frac{d\Psi(\xi)}{d\xi} = 0$	-
Clamped	$V(\xi) = 0$	$\Psi(\xi) = 0$	-	-
Free	-	-	$\frac{d\Psi(\xi)}{d\xi} = 0$	$\kappa GA \left(\frac{\partial V(\xi)}{\partial \xi L} - \Psi(\xi) \right) + G_p \frac{\partial V(\xi)}{\partial \xi} = 0$

4.1 Hinged-Hinged

Figure 3(a) illustrates the relative difference between the frequencies of a hinged-hinged beam on a Pasternak foundation over the frequency of a beam without foundation ($e = 0$, $p = 0$) for various values of the parameter e and p for the first five frequencies. This relative difference delineates the increase in the natural frequencies for a beam on Pasternak foundation with increasing elastic and shear layer stiffness, which represents a constant increase in the foundation general stiffness. As the foundation general stiffness increases, the natural frequencies have a huge increase. The reduction in the increase as the mode number rise presented in Tab. 1 can also be noted.

Figure 3(b) shows the normal modes for the first five frequencies for a beam without foundation and a beam on a Pasternak foundation ($e = 20$, $p = 20$). The figure shows that the presence of the foundation is lower in the normal modes than in the natural frequencies for a hinged-hinged beam. Also, the lower modes presents no remarkable difference from the modes for a beam without foundation, however, the fourth and fifth modes presents a slight difference.

This observation is verified with the relative difference between the amplitude of the normal modes for a beam on a Pasternak foundation and a beam without foundation, as presented in figure 4. The figure shows that the normal modes amplitudes decreases as the foundation general stiffness rise, which represents a contrary behaviour from the present by the frequency parameters.



(a) Increase in the natural frequencies.
(b) First five normal modes.

Figure 3. Pasternak foundation influence on a hinged-hinged beam.

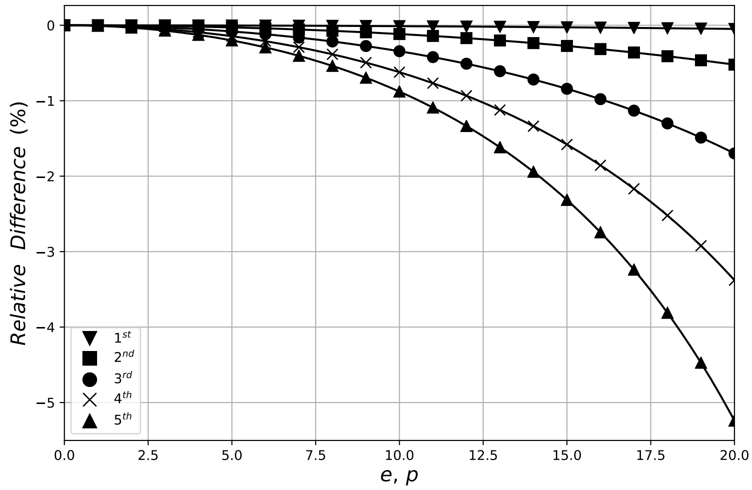


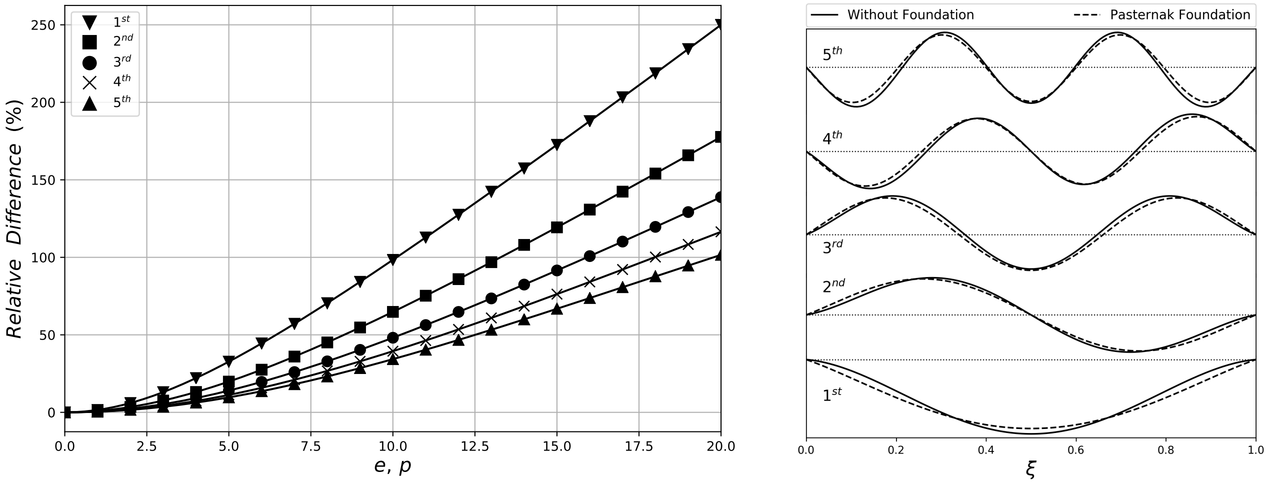
Figure 4. Relative difference between the maximum amplitude of a beam on a Pasternak foundation and a beam without foundation for the hinged-hinged boundary condition.

4.2 Clamped-Clamped

The relative difference between the frequency of a clamped-clamped beam on Pasternak foundation over a beam without foundation ($e = 0$, $p = 0$) for various values of the parameter e and p for the first five frequencies is shown in Fig. 5(a). Even though the frequencies still have the huge increase, it is not high as that presented by the hinged-hinged case, as well as the difference between the increase in the first frequency parameter and the other.

The normal modes presents a remarkable difference when the foundation is present, as shown in Fig. 5(b). The difference between the hinged-hinged and the clamped-clamped boundary condition is the term representing the total rotation introduced in the later, as shown in Tab. 2. Once this boundary condition in inserted, the frequency parameters increase is lower and the difference between the normal modes becomes more significant, when compared to the hinge-hinged case. The clamped-end condition indicates that the foundation even though being proportional to the displacement only has as some influence of the beam total rotation.

Figure 6 presents the reduction in the amplitude of the normal modes for a beam on a Pasternak foundation when compared with a beam without foundation as the foundation general stiffness increases. The figure shows that the reduction in the normal modes amplitude does not present a regular behaviour as presented by the hinged-hinged case. The first and the fifth modes present the major reduction with the fifth mode presenting the major reduction when the beam is on very stiff foundations, even though the first mode present the major reduction for the majority of the foundation general stiffness range.



(a) Increase in the natural frequencies.

(b) First five normal modes.

Figure 5. Pasternak foundation influence on a clamped-clamped beam.

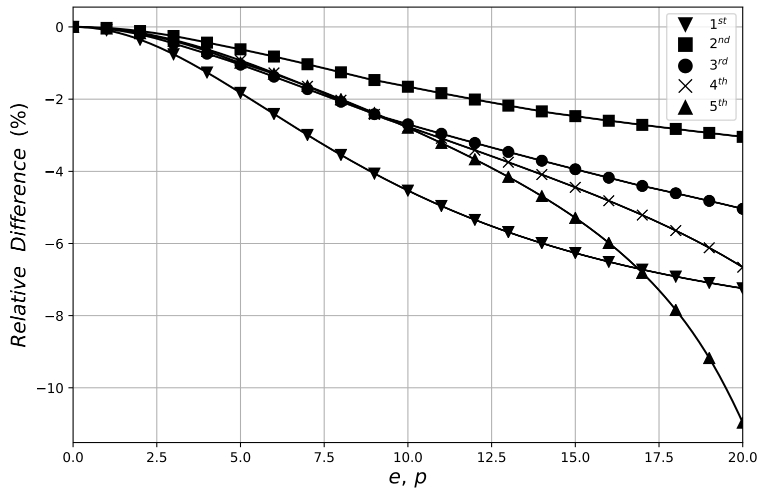


Figure 6. Relative difference between the maximum amplitude of a beam on a Pasternak foundation and a beam without foundation for the clamped-clamped boundary condition.

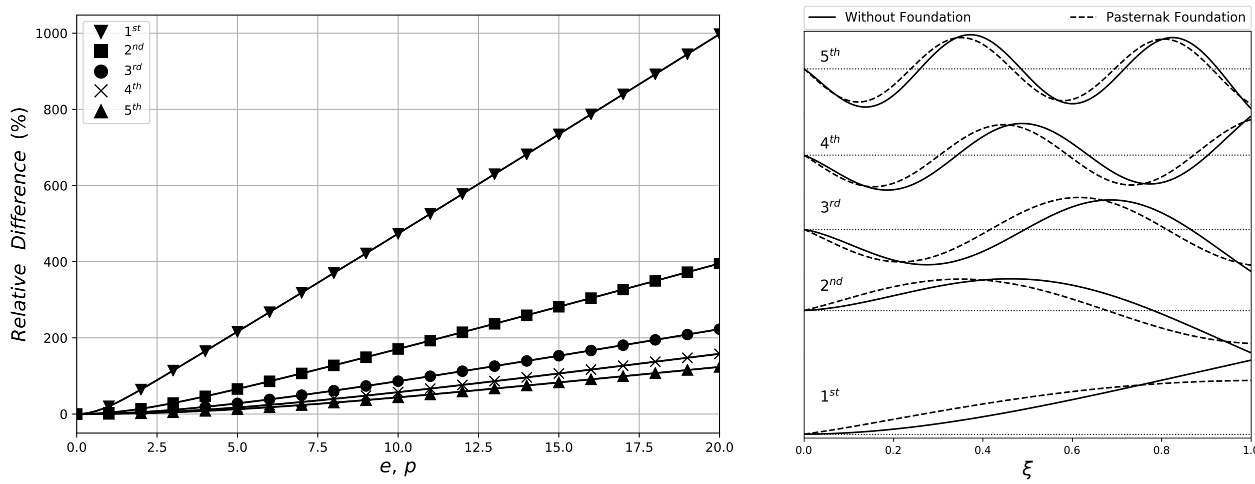
4.3 Clamped-Free

In contrast to observed in the hinged-hinged and clamped-clamped, the clamped-free beam presents a major increase, even higher than that from hinged-hinged. The first mode presents the great increase when compared to the others, ratifying this characteristics of the foundation to affect only the lower modes and the range of influence increasing as the foundation general stiffness rise.

Figure 7(b) shows the effect of the Pasternak foundation in the normal modes of a clamped-free beam. This boundary conditions presents the greater difference in the normal modes with a significant change in all modes. The major reduction in the amplitude, as shown in Fig. 8, is presented by the first mode.

This behavior differs from the presented in the hinged-hinged case in which the major reduction is shown in the fifth mode. The extra factor that the Pasternak foundation insert in the free-end boundary condition, as seen in Tab. 2, appears to influence this different behaviour. As the general stiffness increases, this extra factor magnitude also increases, which the condition accompanying it, $\dfrac{dV(\xi)}{d\xi}$, becomes the significant factor in the expression. As this condition is the same that represents the total rotation of the cross-section of the beam, this result ratifies the assumption that the rotation of the cross-section influences the effect of the foundation.

Also, for a range of the foundation general stiffness, the amplitude of the normal modes, except for the first mode, presents a increase and not a reduction as expected.



(a) Increase in the natural frequencies. (b) First five normal modes.

Figure 7. Pasternak foundation influence on a clamped-free beam.

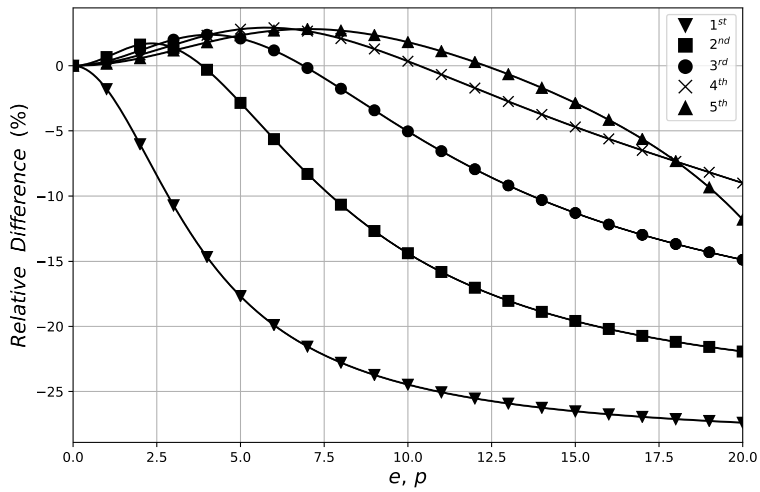


Figure 8. Relative difference between the maximum amplitude of a beam on a Pasternak foundation and a beam without foundation for the clamped-free boundary condition.

5. CONCLUSIONS

This paper presented a finite element method for free vibration analysis of Timoshenko beams on Pasternak foundation regarding the foundation parameters and the boundary conditions. The finite element was developed using cubic and quadratic polynomials for transverse displacement and slope, respectively, for a two-node beam element with two degrees of freedom per node. The finite element and the analytic solutions are compared and discussed with some numerical examples. The investigation determined that the presence of a foundation increase the natural frequencies of the beam vibration. The Pasternak foundation presents higher frequencies than the Winkler foundation, and the increase is expressive for all modes. Also, the presence of the foundation present a reduction in the normal modes. As the foundation stiffness increases, the natural frequency increases and the normal modes amplitude decreases, with the clamped-free case the most affected. For some general foundation stiffness values, the foundation presence increase the normal modes amplitude.

6. REFERENCES

Avramidis, I.E. and Morfidis, K., 2006. “Bending of beams on three-parameter elastic foundation”. *International Journal of Solids and Structures*, Vol. 43, pp. 357–375.

De Rosa, M.A., 1995. “Free vibrations of timoshenko beams”. *Computers and Structures*, Vol. 57, No. 1, pp. 151–156.

El-Mously, M., 1999. “Fundamental frequencies of timoshenko beams mounted on pasternak foundation”. *Journal of Sound and Vibration*, Vol. 228, No. 2, pp. 452–457.

Filonenko-Borodich, M.M., 1940. "Some approximate theories of elastic foundation (in russian)". *Uchenyie Zapiski Moskovskogo Gosudarstvennogo Universiteta, Mekhanica*, Vol. 46, pp. 3–18.

Ghannadiasl, A. and Mofid, M., 2015. "An analytical solution for free vibration of elastically restrained timoshenko beam on an arbitrary variable winkler foundation and under axial load". *Latin American Journal of Solids and Structures*, Vol. 12, pp. 2417–2438.

Hetenyi, M., 1946. *Beams on Elastic Foundation*. The University of Michigan Press.

Kerr, A.D., 1964. "Elastic and viscoelastic foundation models". *Journal of Applied Mechanics*, Vol. 31, No. 3, pp. 491–498.

Kerr, A.D., 1965. "Study of a new foundation model". *Acta Mechanica*, Vol. 1, No. 2, pp. 135–147.

Pasternak, P., 1954. "On a new method of analysis of an elastic foundation by means of two foundation constants (in russian)". *Gosudarstvenno Izdatefslvo Literaturi po Stroitelstvu i Arkhitekture*, Vol. 21.

Soares, A. and Hoefel, S.S., 2015. "Modal analysis for free vibration of four beam theories". In *Proceedings of the 23rd International Congress of Mechanical Engineering*. Rio de Janeiro, Brazil.

Thambiratnam, D. and Zhuge, Y., 1996. "Free vibration analysis of beams on elastic foundation". *Computers and Structures*, Vol. 60, No. 6, pp. 971–980.

Timoshenko, S.P., 1921. "On the correction for shear of the differential equation for transverse vibration of prismatic bars". *Philosophical Magazine*, Vol. 41, pp. 744–746.

Wang, T.M. and Gagnon, L.W., 1978. "Vibrations of continuous timoshenko beams on winkler-pasternak foundations". *Journal of Sound and Vibration*, Vol. 59, No. 2, pp. 211–220.

Wang, T.M. and Stephens, J.E., 1977. "Natural frequencies of timoshenko beams on pasternak foundations". *Journal of Sound and Vibration*, Vol. 51, pp. 149–155.

Winkler, E., 1867. *Die Lehre Von Der Elasticitaet und Festigkeit*. Dominicius, Prag.

Yokoyama, T., 1987. "Vibrations and transient responses of timoshenko beams resting on elastic foundations". *Ingenieur-Archiv*, Vol. 57, pp. 81–90.

Yokoyama, T., 1996. "Vibration analysis of timoshenko beam-columns on two-parameter elastic foundations". *Computers and Structures*, Vol. 61, No. 6, pp. 995–1007.

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