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ABSOLUTE INSTABILITY OF MIXED CONVECTION IN A HEATED HORIZONTAL POROUS CHANNEL

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Abstract. *The present paper employs a linear stability analysis with the purpose to investigate the transition from a convective to an absolute instability in a Darcy-Benard problem. The problem studied is a horizontal throughflow in a porous layer heated from below by an isoflux and internally by a heat source, and with a third kind boundary condition for the temperature at the top. A modal analysis is performed leading to an eigenvalue problem, which allow one to determine critical values for the Rayleigh number as a function of the Peclet number, the wavenumber, the time frequency and the internal heat source. The zero group velocity condition is applied in order to obtain the transition to the absolute instability. The problem is solved using the Generalized Integral Transform Technique (GITT).*

Keywords: *natural convection, absolute instability, porous medium, integral transform.*

1. INTRODUCTION

Convection in porous media represents an important field in the study of heat and mass transfer. Some examples of application can be found in Bejan (1995); Rees (2000); Straughan (2008); Barletta (2011), as well as in chapter 6 of the book by Nield and Bejan (2013). Several studies on the onset of convective instability are available in the literature. The classical Rayleigh-Benard problem studied the onset of the natural convection in a motionless state of fluid open to the atmosphere at the upper boundary. A similar analysis but considering a porous media saturated by a motionless state of fluid has been done by Horton and Rogers Jr (1945) and Lapwood (1948). Prats (1966) investigated the effects of a basic flow in the critical conditions for the onset of convective roll instabilities. Sphaier and Barletta (2014) investigated the mixed convection in a porous media heated from below by an isoflux. Celli *et al.* (2016b) continued the work done by Sphaier and Barletta and studied the convection in a porous media saturated by a fluid flow heated from below with an internal heat generation. Hirata and Ouarzazi (2010) studied the nature of unstable three-dimensional disturbances of viscoelastic flow convection in a porous medium with horizontal through-flow and vertical temperature gradient, they perform a temporal and a spatiotemporal stability analysis of the problem.

This problem is an extension of Braga *et al.* (2017), that analysed the onset of the convective instability in a fluid flow through a porous media heated from below by an isoflux and internally by a heat source, and with a third kind boundary condition for the temperature at the top. The contribution will be so investigate the transition for absolute instability. The system of differential equations are solved using the Generalized Integral Transform Technique (GITT) (Cotta, 1990, 1993). With the GITT, the complex eigenproblem arising from the stability analysis is solved using expansions in terms of simpler eigenproblems with analytical solution. The method transforms the ODE eigenproblem into an algebraic eigenvalue problem that allows the direct calculation of the eigenvalues.

1.1 Problem Description

The governing equations for the considered problem are written in a dimensionless form by defining the following quantities:

$$(x,y,z) = \frac{(x^*,y^*,z^*)}{H}, \quad t = \frac{t^*}{\sigma H^2/\alpha}, \quad \vec{u} = \frac{\vec{u}^*}{\alpha/H}, \quad T = \frac{T^* - T_s}{\dot{q}_0'' H/k}, \tag{1a}$$

$$\mathrm{Bi} = \frac{hH}{k}, \quad \mathrm{Ra} = \frac{g\beta\dot{q}_0'' K H^2}{\nu\alpha k}, \quad \mathrm{Q} = \frac{\dot{q}''' H}{\dot{q}_0''}, \tag{1b}$$

where starred quantities denote dimensional values, H is the layer thickness, α is the average thermal diffusivity, and σ is the ratio between the average volumetric heat capacity of the porous medium $(\rho c)_m$ and the volumetric heat capacity of the fluid $(\rho c)_f$. Therefore, the governing equations are given by

$$\vec{\nabla} \cdot \vec{u} = 0, \tag{2a}$$
$$\vec{\nabla} \times \vec{u} = \mathrm{Ra}\, \vec{\nabla} \times (T \vec{e}_z), \tag{2b}$$
$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \nabla^2 T + \mathrm{Q}, \tag{2c}$$
$$z = 0: \qquad w = 0, \quad \frac{\partial T}{\partial z} = -1, \tag{2d}$$
$$z = 1: \qquad w = 0, \quad \frac{\partial T}{\partial z} + \mathrm{Bi}\, T = 0. \tag{2e}$$

Here, Ra is the Rayleigh number, Bi is the Biot number, and Q is the dimensionless heat source strength.

2. Linear Stability Analysis

2.1 Basic Solution

A stationary solution describing a horizontal flow in the layer can be obtained as

$$\vec{u}_b = (\mathrm{Pe}, 0, 0), \quad T_b = \frac{\mathrm{Q}}{2} + 1 + \frac{\mathrm{Q}+1}{\mathrm{Bi}} - z - \frac{\mathrm{Q}}{2} z^2, \tag{3}$$

where the subscript b stands for *basic flow*, and Pe is an arbitrary real constant playing the role of the PÃl'clet number associated with the basic flow. We note that this solution is valid only if $\mathrm{Bi} > 0$, while it becomes infinite in the limit $\mathrm{Bi} \to 0$. In this limit, the upper wall becomes adiabatic and the correct basic solution is two-dimensional. The solution in the limit $\mathrm{Bi} \to 0$ is that reported in Celli *et al.* (2016a)

2.2 Linearization

Basic state in the system of equations (2) is now perturbed to investigate its linear stability. The introduced disturbances are considered small and written as:

$$T \to T_b + \epsilon\, T_p, \tag{4a}$$
$$u \to u_b + \epsilon\, u_p, \tag{4b}$$
$$v \to v_b + \epsilon\, v_p, \tag{4c}$$
$$w \to w_b + \epsilon\, w_p, \tag{4d}$$

where ϵ is the disturbance amplitude parameter. Since $\epsilon \ll 1$, nonlinear contributions in the governing equations are negligible. After linearization the system of equations becomes:

$$\frac{\partial u_p}{\partial x} + \frac{\partial v_p}{\partial y} + \frac{\partial w_p}{\partial z} = 0, \tag{5a}$$
$$\frac{\partial w_p}{\partial y} - \frac{\partial v_p}{\partial z} = \mathrm{Ra} \frac{\partial T_p}{\partial y}, \tag{5b}$$
$$\frac{\partial u_p}{\partial z} - \frac{\partial w_p}{\partial x} = -\mathrm{Ra}\, \frac{\partial T_p}{\partial x}, \tag{5c}$$
$$\frac{\partial v_p}{\partial x} - \frac{\partial u_p}{\partial y} = 0, \tag{5d}$$
$$\mathrm{Pe}\, \frac{\partial T_p}{\partial x} - \mathrm{Q}\, w_p\, z + \frac{\partial T_p}{\partial t} - w_p = \frac{\partial^2 T_p}{\partial x^2} + \frac{\partial^2 T_p}{\partial y^2} + \frac{\partial^2 T_p}{\partial z^2}, \tag{5e}$$

2.3 Normal Modes

The normal modes method is now employed to simplify equation (5). Assuming that the normal modes are plane waves propagating along the x and y axes, these are written as:

$$T_p \to \hat{T}(z)\, \mathrm{e}^{\mathrm{i}\,(\alpha\, x + \beta\, y - \omega\, t)} \tag{6a}$$
$$u_p \to \hat{u}(z)\, \mathrm{e}^{\mathrm{i}\,(\alpha\, x + \beta\, y - \omega\, t)} \tag{6b}$$
$$v_p \to \hat{v}(z)\, \mathrm{e}^{\mathrm{i}\,(\alpha\, x + \beta\, y - \omega\, t)} \tag{6c}$$
$$w_p \to \hat{w}(z)\, \mathrm{e}^{\mathrm{i}\,(\alpha\, x + \beta\, y - \omega\, t)}, \tag{6d}$$

where α and β are real parameters representing the wavenumber, ω is the angular frequency, while $\hat{T}(z)$, $\hat{u}(z)$, $\hat{v}(z)$ and $\hat{w}(z)$ are amplitudes of the disturbances. After substituting the normal modes in the linearized equations (5) and employing algebraic manipulation, the dispersion equations are obtained:

$$\mathrm{Ra}\,\hat{T}(z)\,(\alpha^2 + \beta^2) \,+\, \hat{w}''(z) \,=\, \hat{w}(z)\,(\alpha^2 + \beta^2) \tag{7a}$$
$$(\mathrm{i}\,\mathrm{Pe}\,\alpha \,+\, \alpha^2 \,+\, \beta^2 \,-\, \mathrm{i}\,\omega)\,\hat{T}(z) \,=\, \hat{w}(z) \,+\, \mathrm{Q}\,z\,\hat{w}(z) \,+\, \hat{T}''(z) \tag{7b}$$

where the boundary conditions are given by:

$$\hat{w}(0) \,=\, 0, \qquad \hat{w}(1) \,=\, 0 \tag{8a}$$
$$\hat{T}'(0) \,=\, 0, \qquad \hat{T}'(1) \,+\, \mathrm{Bi}\,\hat{T}(1) \,=\, 0 \tag{8b}$$

These equations constitute a complex eigenvalue problem, whose solution will be sought via integral transformation in the next sections.

3. Integral Transform Solution of Eigenvalue Problem

This section presents the solution for the dispersion equation (7). The solution is carried-out by means of the Generalized Integral Transform Techinique **?**. The solution process is started by defining the following integral transform pairs:

$$\hat{w}(z) \,=\, \sum_{n=1}^{\infty} \frac{\bar{w}_n\,\Lambda_n(z)}{N_n}, \qquad \bar{w}_n \,=\, \int_0^1 \hat{w}(z)\,\Lambda_n(z)\mathrm{d}z \tag{9a}$$
$$\hat{T}(z) \,=\, \sum_{n=1}^{\infty} \frac{\bar{T}_n\,\Omega_n(z)}{\widehat{N}_n}, \qquad \bar{T}_n \,=\, \int_0^1 \hat{T}(z)\,\Omega_n(z)\mathrm{d}z \tag{9b}$$

in which the eigenfunctions Ω_n and Λ_n are orthogonal solutions to the following Sturm-Liouville type problems:

$$\Lambda'' + \mu^2\,\Lambda \,=\, 0, \tag{10a}$$
$$\Omega'' + \nu^2\,\Omega \,=\, 0, \tag{10b}$$
$$\Lambda(0) \,=\, 0 \qquad \Lambda(1) \,=\, 0 \tag{10c}$$
$$\Omega'(0) \,=\, 0 \qquad \Omega'(1) \,+\, \mathrm{Bi}\,\Omega(1) \,=\, 0 \tag{10d}$$

Multiplying equations (7a) and (7b) respectively by Λ_n and Ω_n, and integrating within $0 \leq z \leq 1$ gives:

$$(\alpha^2 + \beta^2)\,(\mathrm{Ra}\,\int_0^1 \hat{T}(z)\,\Lambda_n(z)\,\mathrm{d}z - \bar{w}_n) \,+\, \int_0^1 \hat{w}(z)\,\Lambda_n''(z)\,\mathrm{d}z \,=\, 0 \tag{11a}$$
$$(\mathrm{i}\,\mathrm{Pe}\,\alpha \,+\, \alpha^2 \,+\, \beta^2 \,-\, \mathrm{i}\,\omega)\,\bar{T}_n \,-\, \int_0^1 \hat{w}(z)\,\Omega_n(z)\,\mathrm{d}z \,-\, \mathrm{Q}\int_0^1 z\,\hat{w}(z)\,\Omega_n(z)\,\mathrm{d}z - \int_0^1 \hat{T}(z)\,\Omega_n''(z)\,\mathrm{d}z = 0 \tag{11b}$$

where integration-by-parts and some simplification was employed.

Then, by using the eigenvalue problem information (eqs. (10)), substituting the inverse formula (eqs. (9a)) gives:

$$(\alpha^2 + \beta^2)\,(\mathrm{Ra}\,\sum_{m=1}^{\infty} A^*_{n,m}\bar{T}_m \,-\, \bar{w}_n) \,-\, \mu_n^2\,\bar{w}_n \,=\, 0 \tag{12a}$$
$$(\mathrm{i}\,\mathrm{Pe}\,\alpha \,+\, \alpha^2 \,+\, \beta^2 \,-\, \mathrm{i}\,\omega)\,\bar{T}_n \,-\, \sum_{m=1}^{\infty} A_{n,m}\,\bar{w}_m \,-\, \mathrm{Q}\sum_{m=1}^{\infty} B_{n,m}\,\bar{w}_m \,+\, \nu_n^2\,\bar{T}_n \,=\, 0 \tag{12b}$$

where the integral coefficients are given by:

$$A^*_{n,m} = \frac{1}{\hat{N}_m}\int_0^1 \Omega_m\,\Lambda_n\,\mathrm{d}z, \tag{13a}$$

$$A_{n,m} = \frac{1}{N_m}\int_0^1 \Lambda_m\,\Omega_n\,\mathrm{d}z, \tag{13b}$$

$$B_{n,m} = \frac{1}{N_m}\int_0^1 z\,\Lambda_m\,\Omega_n\,\mathrm{d}z, \tag{13c}$$

After truncating the infinite series representation to a finite order $n_{\max}$, system (12) can be written in a vector form:

$$(\alpha^2+\beta^2)\,(\mathrm{Ra}\,\boldsymbol{A}^*\,\bar{\boldsymbol{T}} - \bar{\boldsymbol{w}}) - \boldsymbol{D}\,\bar{\boldsymbol{w}} = \boldsymbol{0} \tag{14a}$$

$$(\mathrm{i}\,\mathrm{Pe}\,\alpha + \alpha^2 + \beta^2 - \mathrm{i}\,\omega)\,\bar{\boldsymbol{T}} - \boldsymbol{A}\,\bar{\boldsymbol{w}} - \mathrm{Q}\,\boldsymbol{B}\,\bar{\boldsymbol{w}} + \boldsymbol{E}\,\bar{\boldsymbol{T}} = \boldsymbol{0} \tag{14b}$$

where $\boldsymbol{D}$ and $\boldsymbol{E}$ are diagonal matrices whose coefficients are given by $D_{n,n} = \mu_n^2$ and $E_{n,n} = \nu_n^2$.

Equation (14a) can be solved for $\bar{\boldsymbol{w}}$, producing:

$$((\alpha^2+\beta^2) + \boldsymbol{D})\,\bar{\boldsymbol{w}} = (\alpha^2+\beta^2)\,\mathrm{Ra}\,\boldsymbol{A}^*\,\bar{\boldsymbol{T}} \tag{15}$$

$$\bar{\boldsymbol{w}} = (\alpha^2+\beta^2)\,\mathrm{Ra}\,\boldsymbol{F}\,\boldsymbol{A}^*\,\bar{\boldsymbol{T}} \tag{16}$$

where $\boldsymbol{F}$ is a diagonal matrix given by

$$F_{n,m} = \frac{\delta_{n,m}}{(\alpha^2+\beta^2) + \mu_n^2} \tag{17}$$

Substituting eq. (16) in (14b) produces

$$(\mathrm{i}\,\mathrm{Pe}\,\alpha + \alpha^2 + \beta^2 - \mathrm{i}\,\omega)\,\bar{\boldsymbol{T}} - (\boldsymbol{A} + \mathrm{Q}\,\boldsymbol{B})\,(\alpha^2+\beta^2)\,\mathrm{Ra}\,\boldsymbol{F}\,\boldsymbol{A}^*\,\bar{\boldsymbol{T}} + \boldsymbol{E}\,\bar{\boldsymbol{T}} = \boldsymbol{0} \tag{18}$$

This equation can finally be written in the following form

$$(\boldsymbol{M} - \mathrm{Ra}\,\boldsymbol{H})\,\bar{\boldsymbol{T}} = \boldsymbol{0} \tag{19}$$

where

$$\boldsymbol{M} = (\mathrm{i}\,\mathrm{Pe}\,\alpha + \alpha^2 + \beta^2 - \mathrm{i}\,\omega)\,\boldsymbol{I} + \boldsymbol{E} \tag{20}$$

$$\boldsymbol{H} = (\boldsymbol{A} + \mathrm{Q}\,\boldsymbol{B})\,(\alpha^2+\beta^2)\,\boldsymbol{F}\,\boldsymbol{A}^* \tag{21}$$

such that Ra can be calculated from the eigenvalues of $\boldsymbol{H}^{-1}\boldsymbol{M}$. However, this is calculated as generalized eigenvalue problem which does not require the inversion of $\boldsymbol{H}$.

3.1 Figures and tables

In this section are presented some results for the onset of absolute instability. According to the mathematical formalism of absolute instabilities, the necessary, but not sufficient, condition for absolute instability is the existence of a saddle point (α ,β) in the two complex α and β planes defined by:

$$D(\omega,\alpha,\beta,Q,Pe,Ra) = 0 \qquad \text{with} \quad \frac{\partial\omega}{\partial\alpha} = \frac{\partial\omega}{\partial\beta} = 0, \tag{22}$$

where $D(\omega,\alpha,\beta,Q,Pe,Ra)$ is the dispersion equation. A detailed analysis on the mathematical formalism of absolute instabilities can be found at Briggs (1964).

The results where focus on the analysis of the transversal modes of instabilities, $\beta = 0$. Figure 1 show the spatial branches behavior for different values of Ra for a fixed Peclet number. This (α_r, α_i)-plane is analyzed to check the possible existence of the pinching point in the complex map to find the critical Rayleigh number that transition to the absolute instability. An shown in Fig 1 the Ra_c (for Pe = 7) is approximately 37.85.

Figure 2 shows the values of critical Rayleigh number as a function of the Peclet number, Pe, for different values of the internal heat source, Q. It can be observed that the increase of the Peclet number also increase the critical Rayleigh number i. e. more energy is necessary to initiate natural convecion. For really big values of Q the problem become almost completely unstable

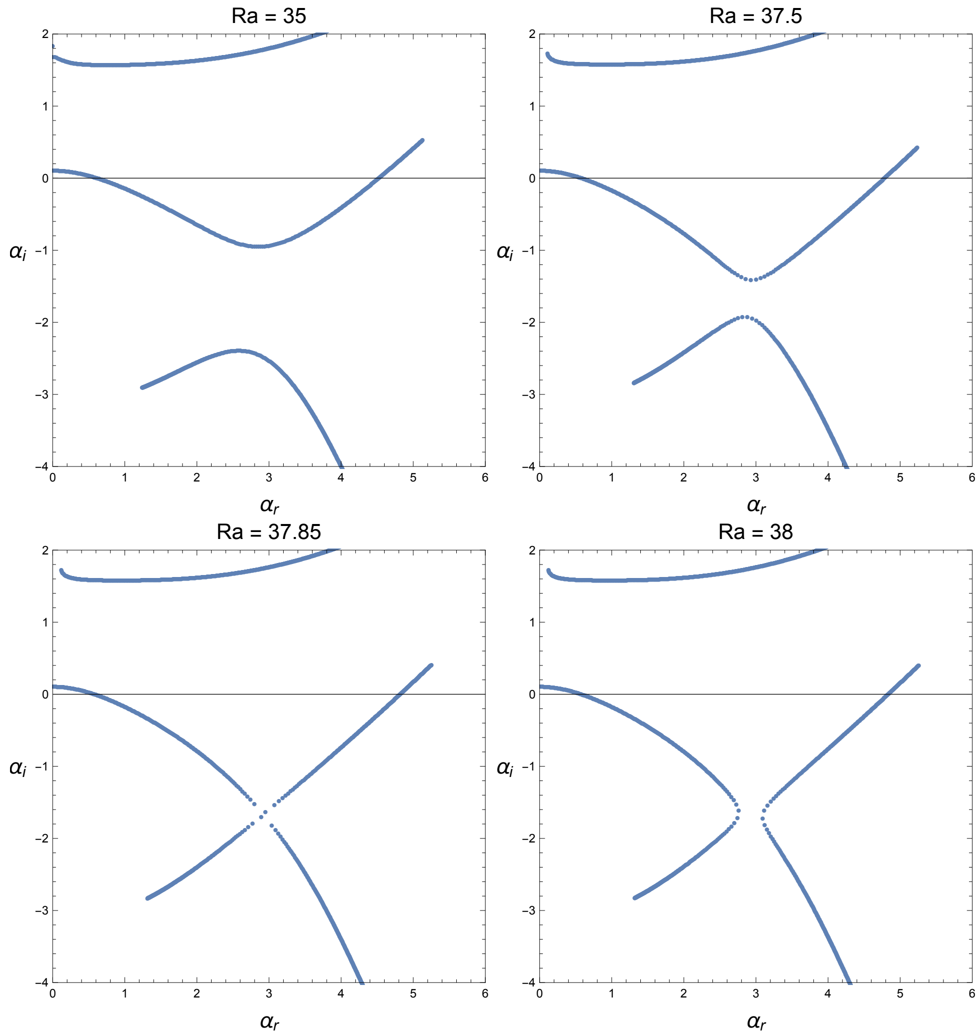

Figure 1. Spatial branches in the complex (α_r, α_i)-plane for different values of Ra, for β = 0, Pe = 7, Biot = 1 and Q = 0.

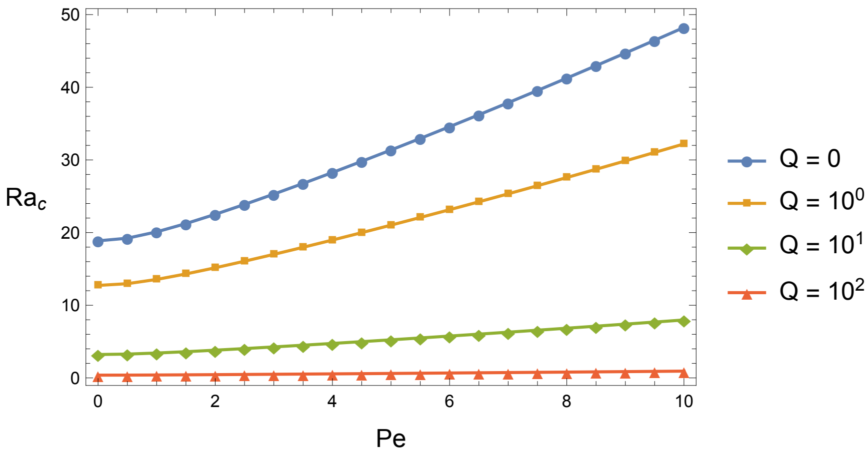

Figure 2. Critical Rayleigh number (R_c) versus Peclet number for different values of Q for Biot = 1 and $\beta = 0$

4. Conclusion

This paper presents a linear stability analysis, on the onset of convection, in a porous channel. The lower wall is heated by a uniform external heat flux and the upper wall loses heat through convection to an external fluid. In this paper only the transversal rolls ($\beta = 0$) were investigated. The solution has been obtained by the GITT method.

An uniform internal heat source is imposed to identify how this parameter influence the critical Rayleigh number, required to induce convection. The conclusions reached are that the increase of the Peclet number makes the problem more stable and the critical values of Rayleigh number decreases as the internal heat source, Q, increases, this means that Q destabilizes the problem. As the internal heat source increase the dependence with Peclet number decrease, and for really big values of Q numbers the problem become almost completely unstable and independent of Pe.

5. ACKNOWLEDGEMENTS

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