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MARANGONI INSTABILITY OF A THIN LIQUID FILM FLOW WITH VISCOUS DISSIPATION

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Abstract. *This paper analyses the linear stability of a thin liquid film flowing over a horizontal plate. This plate is considered impermeable and adiabatic. The upper surface of the film is considered to be a free boundary with a surface tension. The thermoconvective instability is triggered by this surface tension. Viscous dissipation within the fluid induces temperature gradients, generating variation in the surface tension, and then initiating the instability phenomena. The linear stability analysis is performed using the normal modes method. A system of four coupled differential equations is generated, defining an eigenvalue problem. The convective instability of longitudinal rolls is analysed.*

Keywords: *LINEAR STABILITY ANALYSIS, SURFACE TENSION, MARANGONI EFFECT*

1. INTRODUCTION

The convective cells were seen firstly by Bénard (1901). This mechanism of instability was driven, according to Rayleigh (Rayleigh, 1916), by the bouyance effect. However, later studies proved that the phenomena observed by Bénard was driven by another mechanism. Block (1956) showed that the convective cells in a thin fluid flow (less than 1mm), such as the one observed by Bénard, could not happen due to bouyance. Rather this instability is triggered by another phemonema: the surface tension. Soon afterwards, Pearson (1958) showed an analytical explanation for the problem. As a matter of fact, both mechanism have been studied simultaneously in order to catch the onset of instability (Reichenbach and Linde (1981)). These studies led to the conclucions that was earlier observed by Bénard (1901) and explained by Rayleigh (Rayleigh, 1916), was apperently driven by surface tension gradients generated at the interface, instead of density gradients (bouyance). This phenomena is known as the Marangoni effect.

Although differents models were developed to model Marangoni convection, this paper considers one of the simplest. It evaluates a single layer of liquid that shares an immiscible and nondeformable interface with a layer of gas whose bulk effects can be considered negligible (Pearson (1958))(Reichenbach and Linde (1981)). This work presents yet another extension for this model, whose onset of Poiseuille-Marangoni convection is driven by the surface tension gradient generated by viscous dissipation. This extension was already carried out considering the lower film surface adiabatic and the upper film surfacce subjected to Robin boundary condition(Celli *et al.* (2015)). The only mechanism responsible to generate heat, and therefore temperature gradient, is the viscous dissipation. Nonetheless, in this paper, finite values of Prandtl is considered, and nonlinear phenomenon are allowed to appear.

2. MATHEMATICAL MODEL

A Newtonian fluid flowing over a inpermeable adiabatic horizontal plate is studied. The upper fluid film surface is considered free boundary subject to temperature surface tension, so the Marangoni effect takes place as a possible cause for the thermoconvective instability. It is constrained by a Robin boundary condition. The viscous dissipation accounts as a internal heat source. The fluid is assumed to have a constant density ρ. Therefore, no bouyance effect is allowed. The governig equation of local mass, linear momentum and energy take, respectively, the following form

$$\nabla.\mathbf{u} = 0 \qquad \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla.\mathbf{u} = -\frac{\nabla P}{\rho} + \nu.\nabla^2\mathbf{u} \qquad \frac{\partial T}{\partial t} + \mathbf{u}.\nabla T = \alpha \nabla^2 T + \frac{2\nu}{c} D_{ij} D_{ij} \tag{1}$$

where the summation over repeated index is implied, $\mathbf{u} = (u, v, w)$ is the velocity vector, t is the time, P is the pressure, ν is the kinematic viscosity, T is the temperature, α is the thermal diffusivity and c is the specific heat. The strain tensor D_{ij} contributes to the heat generated by the viscous dissipation, and it assumes the following form

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2}$$

where $\mathbf{x} = (x, y, z)$ is the position vector expressed by Cartesian coordinates. The boundary condition at the free film surface are expressed in terms of the shear stress and the surface tension gradient

$$\tau_{zx} = \frac{\partial S}{\partial x} \quad and \quad \tau_{zy} = \frac{\partial S}{\partial y}, \quad for \quad z = H \tag{3}$$

where τ is the shear stress, H is the film thickness and S is the surface tension, which is assumed to be a linear function of the temperature, i.e, $S = S_0 - \sigma(T - T_0)$. The symbol σ is a parameter defined as a negative surface tension variation with temperature $-\partial S/\partial T$. As the plate is considered as impermeable and the interface nondeformable, the boundary conditions are the following.

$$for \quad z = 0 \tag{4a}$$

$$u = v = w = 0, \quad \frac{\partial T}{\partial z} = 0 \tag{4b}$$

$$for \quad z = 1 \tag{4c}$$

$$\frac{\partial u}{\partial z} = -\frac{\sigma}{\mu}\frac{\partial T}{\partial x}, \quad \frac{\partial v}{\partial z} = -\frac{\sigma}{\mu}\frac{\partial T}{\partial y}, \quad w = 0, \quad \frac{\partial T}{\partial z} + \frac{h}{k}(T - T_\infty) = 0 \tag{4d}$$

where μ is the dynamic viscosity, h is the external heat transfer coefficient, k is the thermal conductivity and T_∞ is the temperature of the fluid above the liquid film at a large distance enough not to be disturbed by the film itself.

This governing equation can be written in a dimensionlesss form using the governing paramters and scaling. The scaling employed to obtain the dimensionless equations is

$$\frac{\alpha}{H^2} \to t, \frac{x}{H} \to x, \frac{H}{\alpha}\mathbf{u} \to \mathbf{u}, \frac{T - T_\infty}{\Delta T} \to T, \frac{H^2}{\mu\alpha}P \to P, \frac{H^2}{\alpha}D_{ij} \to D_{ij}, H\nabla \to \nabla \tag{5}$$

and the problem governing parameters are

$$Bi = \frac{Hh}{k}, \quad Pr = \frac{\nu}{\alpha}, \quad Ma = \frac{\alpha H \Delta T}{\alpha\mu} \tag{6}$$

where Bi is the Biot, Pr is the Prandtl and Ma is the Marangoni number, respectively.

Substituting the scaling Eq.5 and the governing parameters Eq.6 into Eq.1, one obtains the set of dimensionless equation

$$\nabla.\mathbf{u} = 0, \qquad \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla.\mathbf{u} = Pr\left(\nabla P + \nabla^2\mathbf{u}\right), \qquad \frac{\partial T}{\partial t} + \mathbf{u}.\nabla T = \nabla^2 T + 2D_{ij}D_{ij} \tag{7}$$

The new boundary conditions in the dimensionless become afterwards

$$for \quad z = 0 \tag{8a}$$

$$u = v = w = 0, \quad \frac{\partial T}{\partial z} = 0 \tag{8b}$$

$$for \quad z = 1 \tag{8c}$$

$$\frac{\partial u}{\partial z} = -Ma\frac{\partial T}{\partial x}, \quad \frac{\partial v}{\partial z} = -Ma\frac{\partial T}{\partial y}, \quad w = 0, \quad \frac{\partial T}{\partial z} + Bi.T = 0 \tag{8d}$$

2.1 STEADY SOLUTION

A steady parallel flow in the (x,y) plane alinged in the x direction is considered. The one responsible for driving the flow is the pressure gradient ∇P and it is assumed to be parallel to the x axis. Hence, the steady solution for the velocity and temperature profile can be determined analytically

$$P_b(x) = Ax + const. \qquad \mathbf{u}_b(z) = \frac{Az}{2}(z-2)\mathbf{n} \qquad T_b(z) = \frac{A^2}{12}\left(3 - 6z^2 + 4z^3 - z^4 + \frac{4}{Bi}\right) \tag{9}$$

where $\mathbf{n} = (1, 0, 0)$ and the subscript b denotates the steady state. The constant A can be written in function of the Péclet number Pe, which is set to be equal to the average value of the basic flow velocity over the film section.

$$Pe = \int_0^1 \mathbf{u}_b \mathbf{n} dz \tag{10}$$

which leads to $A = -3Pe$. As one might notice, the flow rate per unit width is equal to the Pe number.

3. LINEAR STABILITY ANALYSIS

In order to perform a linear stability analysis, the velocity, pressure and temperature fields are redefined, in order to compose the basic solution and the sum of small perturbation, namely,

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_b(z) + \epsilon \mathbf{U}_p(z) e^{i(\alpha x + \beta y - \omega t)} \\ P &= P_b(x, y) + \epsilon \psi_p(z) e^{i(\alpha x + \beta y - \omega t)} \\ T &= T_b(z) + \epsilon \Theta_p(z) e^{i(\alpha x + \beta y - \omega t)} \end{aligned} \tag{11}$$

where $\mathbf{U}_p = (up(z), vp(z), w_p(z))$, $\Psi_p(z)$ and $\Theta_p(z)$ are the velocity, pressure and temperature pertubation, respectively; α is the wave number in the x direction; β is the wave number in the y direction; ω is the frequency and ϵ is a constant assumed to be small enough to neglect the contributions of the terms $O(\epsilon^2)$, i.e., the nonlinear terms of the disturbance. After substituting Eq.11 into Eq.7, one obtains a system of ODEs, characterizing an eigenvalue problem.

$$i(\alpha u_p + \beta v_p) + w_p{}' = 0, \tag{12a}$$
$$(Pr(\alpha^2 + \beta^2) - i\omega + i\alpha u_b)u_p + iPr\alpha\psi_p + w_p u_b{}' - Pru_p{}'' = 0, \tag{12b}$$
$$(Pr(\alpha^2 + \beta^2) - i\omega + i\alpha u_b)v_p + iPr\beta\psi_p - Prv_p{}'' = 0, \tag{12c}$$
$$(Pr(\alpha^2 + \beta^2) - i\omega + i\alpha u_b)w_p(z) + Pr(\psi_p{}' - w_p{}'') = 0, \tag{12d}$$
$$(\alpha^2 + \beta^2 - i\omega + i\alpha u_b)\Theta_p + w_p(T_b{}' + -2i\alpha u_b{}') - 2u_b{}'u_p{}' - \Theta_p{}'' = 0 \tag{12e}$$

This system of equatios above cannot be solved because the boundary conditions of pressure is unknown. One can overcome this problem by invoking Orr-Sommerfeld equation. Moreover, i) to ease the search for the problem eigenvalues, the second derivative of Eq.12a was obtainded, ii) Eq.12b and Eq.12c were combined into a new equation. Finnaly, one obtains the system of equations which can be solved numerically to find the eigenvalues and eigenvectors of the present problem.

$$i(\alpha u_p{}'' + \beta v_p{}'') + w_p{}''' = 0, \tag{13a}$$
$$\alpha(Pr(\alpha^2 + \beta^2) - i\omega + i\alpha u_b) - \beta(Pr(\alpha^2 + \beta^2) - i\omega + i\alpha u_b) - \beta w_p u_b' + Pr(\beta u_p'' - \alpha v_p'') = 0 \tag{13b}$$
$$(-(\alpha^2 + \beta^2)(Pr(\alpha^2 + \beta^2) - i\omega + i\alpha u_b)w_p - i\alpha u_b'') + (2Pr(\alpha^2 + \beta^2) - i\omega + i\alpha u_b(z))w_p'' - Prw_p^{(iv)} = 0 \tag{13c}$$
$$(\alpha^2 + \beta^2 - i\omega + i\alpha u_b)\Theta_p + w_p(T_b{}'(z) + -2i\alpha u_b{}') - 2u_b{}'u_p{}' - \Theta_p{}'' = 0 \tag{13d}$$

Two more boundary conditions for w_p can be obtained from Eq.12a. Afterall, the new set of boundary conditions yields

$$for \quad z = 0 \tag{14a}$$
$$u_p = v_p = w_p = w_p' = 0, \quad \Theta_p = 0 \tag{14b}$$
$$for \quad z = 1 \tag{14c}$$
$$u_p{}' = -i\alpha Ma\Theta_p, \tag{14d}$$
$$v_p{}' = -i\beta Ma\Theta_p, \tag{14e}$$
$$w_p = w_p' = 0, \tag{14f}$$
$$\Theta_p{}' + Bi\Theta_p = 0 \tag{14g}$$

This system of equations was solved numerically and implemented with the software *Mathematica 10* using the built-in functions *NDSolve* and *FindRoot*.

4. RESULTS AND DISCUSSION

The longitudinal rolls are the ones whose axes are parallel to the steady flow. These modes can be obtained fixing $\alpha = 0$ and allowing β to change. Additionally, one can reduce the number of governing parameters in the problem by performing the following change of variables

$$\Theta_p^* = \frac{\Theta_p}{Pe^2} \qquad u_p^* = \frac{u_p}{Pe} \tag{15}$$

Afterwards, Pe and Ma can be joined into a new parameter, namely $\Lambda = Ma.Pe^2$. Moreover, according to the Exchange of Instabilities (Davis, 1969), ω can be set to zero in Eq.13.

Fig.1 displays the marginal curve for different values of Bi and Pr.

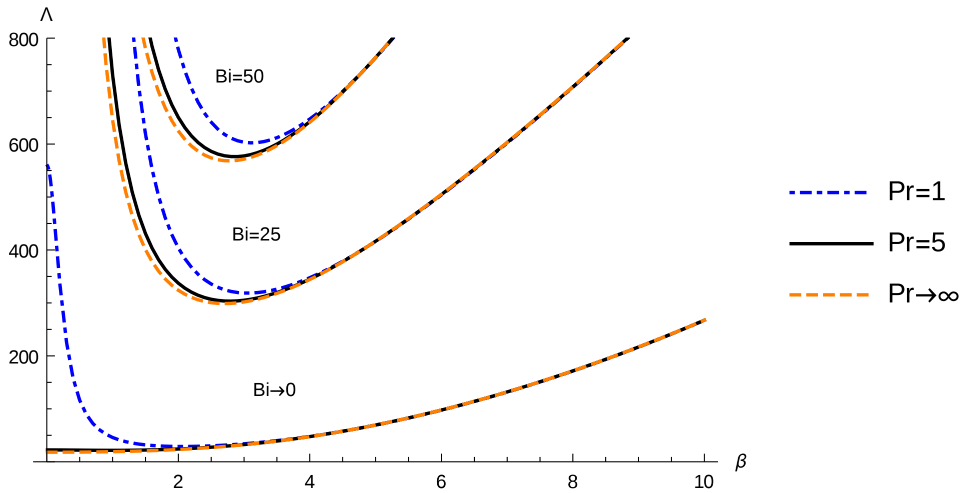

Figure 1. Marginal stability curve for the longitudinal modes, for $Bi = 0$, $Bi = 25$ and $Bi = 50$.

It might be noticed that a change in Pr does not produce significant effect on the marginal curve shape for high values of Bi. Additionally, the marginal curves have always an upward concave shape. The only exception takes place when $Bi \to 0$ and Pr is not small enough. In this situation, the minimum value of Ma is obtained with $\beta \to 0$, which is an uniform case. It can be seen from Eq.9 that in the limiting case where $Bi \to 0$, the steady temperature profile becomes singular. Physically, as both the lower and upper layer are adiabatic and there is heat generation within the fluid, the steady temperature profile cannot vary only in the z direction.

It is also observed in Fig.1 that the curves tend to jump to infinity as $Bi \to \infty$ for any value of Pr. It turn out that the flow is unconditionaly stable when the upper layer is kept isothermal.

In order to take the critical value for β and Ma, the lowest point of the marginal curve was taken for various different values of Bi. Fig.2 and Fig.3 display the critcal values of β and Ma for various Bi, respectively.

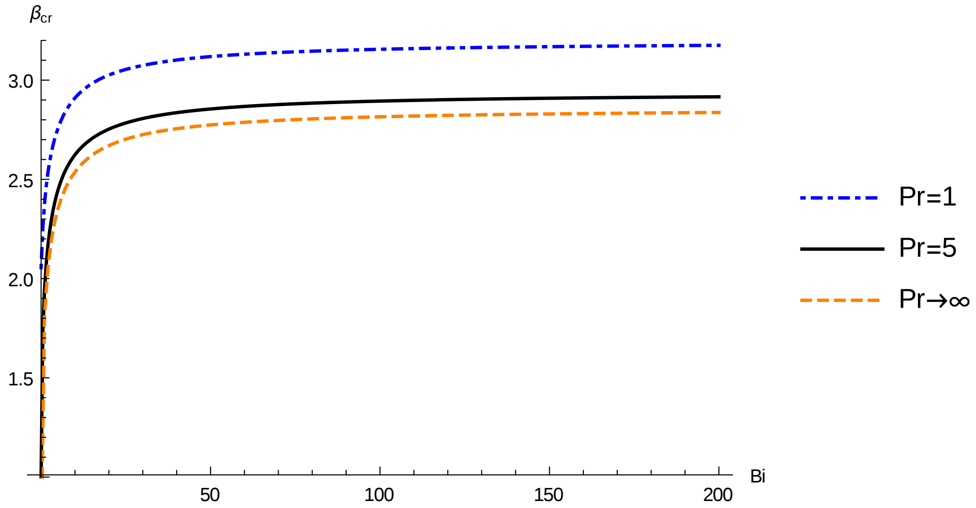

Figure 2. Critical values of β in function of Bi, for $Pr = 1$ (blue dashdotted), $Pr = 5$ (black) and $Pr \to \infty$ (orange dashed).

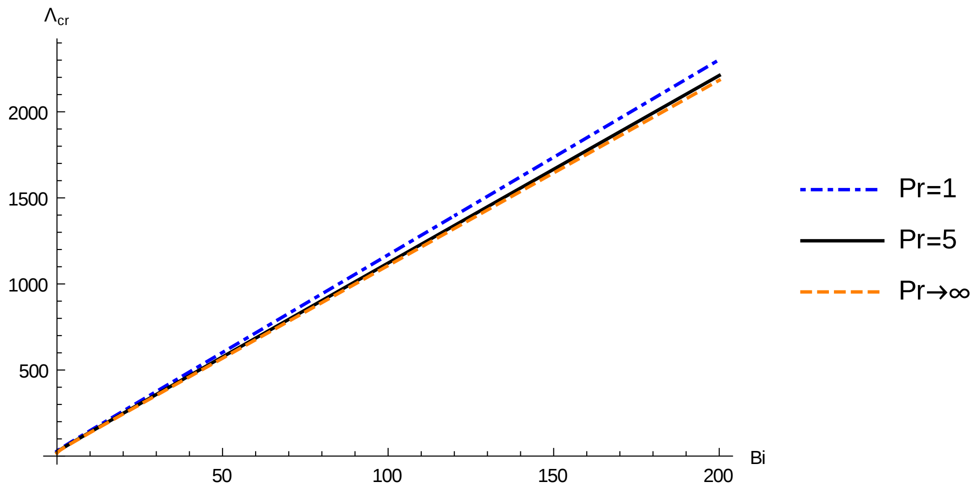

Figure 3. Critical values of Λ in function of Bi, for $Pr = 1$ (blue dashdotted), $Pr = 5$ (black) and $Pr \to \infty$ (orange dashed).

Fig.2 shows β_{cr} for all different Pr tends asymptotically to a constant value for high values of Bi. As for the cases considered, $\beta_{cr} \to 3.196$ for $Pr = 1$, $\beta_{cr} \to 2.938$ for $Pr = 5$ and $\beta_{cr} \to 2.859$ for $Pr \to \infty$.

On the other hand, for low values of Bi, β_{cr} tends to a constant value as Bi approaches zero, i.e., the upper film becomes adiabatic. As already mentioned, in the case $Pr \to \infty$ and $Bi \to 0$, the $\beta_{cr} \to 0$. But for any $Pr > 0$, the β_{cr} is a finite number when $Bi \to 0$. For the cases considered, $\beta_{cr} \to 1.00089$ for $Pr = 1$ and $\beta_{cr} \to 2.05549$ for $Pr = 5$ as $Bi \to 0$.

As for the Ma_{cr}, Fig.3 shows a quasi-linear growth as Bi increases. It leads to the conclusion that in the limiting case $Bi \to \infty$, the value of $Ma_{cr} \to \infty$ as well. In this situation, the problem is considered stable.

It must be pointed out that when $Pr \to 0$, both viscosity and pressure drop off the momentum equation. It means there is no viscous dissipation within the fluid, and therefore no temperature gradient can be expected. As the Marangoni effect takes place only when there is temperature gradient on the surface, it becomes clear that the problem is stable when $Pr \to 0$. However, such low values for Pr are not commonly encountered in general applications. One of the few excpetions are liquid metal flow, wich can reach values of $O(10^{-3})$ (Mills, 1999).

5. CONCLUSION

A thin liquid flow flowing over a horizontal adiabatic place was analysed. The upper surface film is subjected to Robin boundary condition. The onset of the instability was predicted using the LSA method. The instability trigger is known as the Marangoni effect, and it takes place whenever there is temperature gradients on the surface. A quite simple Marangoni model was used to model the surface tension. A system of four differential equation for the pertubation motion was obtained for the longitudinal modes. This system was solved numerically by means of the software *Mathematica 10*.

The marginal curves for the longitudinal rolls were displayed for different values of Bi and Pr. The increasing in the Bi tends to move upwards the marginal curve. A change in Pr does not produce significant effect on the marginal curve shaped for values of Pr not to small.

The critical value of β for the onset of convective instability is a monotonically increasing function of the Biot number. When $Bi \to \infty$, the critical value of Λ tends to infinity. This finding is consistent with the expected stability of the steady flow when the upper free surface is constrained to be perfectly isothermal. In this scenario, no gradient of temperature can be developed within the fluid.

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