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DYNAMIC ANALYSIS OF A VISCOELASTIC TIMOSHENKO BEAM

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Abstract. *Viscoelastic materials are widely used for passive damping in a variety of engineering structures due to the need for structural stability and durability. Beams are often used as structural elements in many of those structures, like rotor blades, transmission shafts, frames and robotic arms. In this paper, the full development and analysis of Timoshenko's beam for transversely vibrations uniform viscoelastic beam are presented for classical boundary conditions. The governing equation of motion is obtained based upon Hamilton's principle and constitutive relations. The viscoelastic beam material is constituted by the Kelvin-Voigt rheological model. Finally, the influence of the viscous internal friction on the natural frequencies and waves dispersion are discussed and numerically demonstrated.*

Keywords: *Timoshenko beam, viscoelasticity, oscillation, internal friction, free vibration*

1. INTRODUCTION

According to Martin (2014), viscoelastic materials have found wide application in engineering structures, due to their ability to dampen out the vibrations. These materials includes: metals at elevated temperatures, rubbers and polymers with both characteristics of elastic and viscous solids. Recently, many researchers demonstrated interest in studying dynamical behavior of viscoelastic beams. Although the classical Euler-Bernoulli theory (EBT) predicts the frequencies of flexural vibration for lower modes of slender beams quite accurately, Timoshenko beam theory (TBT) results were more accurate for higher modes or non slender beams where the effects of shear deformation and rotatory inertia become more significant.

Usually, two different models of the beam material, denoted as Kelvin-Voigt and Maxwell, are used to describe the viscoelastic behavior (Marynowski, 2002). The first model consists of a parallel association of a spring and a dash-pot, while the second is constituted by a series association of these mechanical elements. Several researchers, (Mandan, 1969; Manevich and Kolakowski, 2011; Lang and Leyendecker, 2016) focused their investigations of TBT dynamics on Kelvin model. Most recently, Chen *et al.* (2017) studied the effects of the length-to-depth ratio, axial tension, and the viscosity coefficients on the natural frequencies on viscoelastic Timoshenko beam-columns constituted by the Kelvin-Voigt model. In their publication, is discussed how vicoelasticity changes slightly natural frequencies and modal functions.

In this paper Kelvin-Voigt material is used to describe the dynamics of viscoelastic TBT beam. The motions equations are derived from Euler-Lagrange equation. The influence of viscous internal friction is investigated on natural frequencies predictions for classical boundary conditions and on dispersion relations of an infinite length beam. Numerical results are compared with literature to signify the difference among viscoelastic and elastic Timoshenko beams.

2. VISCOELASTIC TIMOSHENKO BEAM FORMULATION

2.1 Governing equations

Timoshenko (1921) proposed a beam model which includes the effects of rotatory inertia and shear distortion in the Euler-Bernoulli beam model. The total kinetic energy is derived partly from the motion of translation and partly from the rotation as follows:

$$T = \frac{1}{2}\left(\int_0^L \rho A \left(\frac{\partial v(x,t)}{\partial t}\right)^2 dx + \int_0^L \rho I \left(\frac{\partial \psi(x,t)}{\partial t}\right)^2 dx\right), \tag{1}$$

in which L is the length of the beam, A, the cross-sectional area, I, the moment of inertia of cross section, ρ, the mass per unit volume, $v(x,t)$, the transverse deflection and $\psi(x,t)$, the bending slope. The potential energy is derived partly from the bending deformation and partly from the shear deformation. Therefore, total potential energy is given by:

$$U = \frac{1}{2}\left(\int_0^L A\,\sigma_x \epsilon_x dx + \int_0^L A\,\tau_{xy}\gamma_{xy} dx\right), \tag{2}$$

where σ_x is the axial stress, ϵ_x, the axial strain, τ_{xy}, shear stress, and γ_{xy}, the shear strain.

Constitutive relations are assumed according to the Voigt law, shown in Fig. 1, for normal and shear stresses as follows (Manevich and Kolakowski, 2011):

$$\sigma_x = E\left(\epsilon_x + \eta_1 \frac{\partial \epsilon_x}{\partial t}\right), \quad \text{and} \quad \tau_{xy} = G\left(\gamma_{xy} + \eta_2 \frac{\partial \gamma_{xy}}{\partial t}\right), \tag{3}$$

where G is the modulus of rigidity, η_1 and η_2 are the coefficients of viscosity. For simplicity, we assume $\eta_1 = \eta_2 = \eta$.

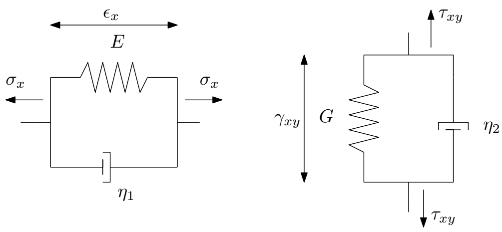

Figure 1. Kelvin-Voigt rheologic model

Equations of motion are obtained using Hamilton's principle:

$$\int_{t_1}^{t_2} \delta(T - U)dt + \int_{t_1}^{t_2} \delta W_{nc} dt = 0, \tag{4}$$

in which δW_{nc} is the virtual work done by nonconservative forces, t_1 and t_2, times at which the configuration of the system is known, and $\delta(\,)$, the symbol denoting virtual change in the quantity in parentheses. Substituting Eqs. (1) and (2) into Eq. (4) and, after some manipulations, we have two decoupled equations expressed as (Mandan, 1969):

$$EI\left(1 + \eta\frac{\partial v}{\partial t}\right)^2 \frac{\partial^4 v}{\partial x^4} - \rho I\left(1 + \frac{E}{KG}\right)\left(1 + \eta\frac{\partial v}{\partial t}\right)\frac{\partial^4 v}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{KG}\frac{\partial^4 v}{\partial t^4} + \rho A\left(1 + \eta\frac{\partial v}{\partial t}\right)\frac{\partial^2 v}{\partial t^2} = 0, \tag{5}$$

$$EI\left(1 + \eta\frac{\partial \psi}{\partial t}\right)^2 \frac{\partial^4 \psi}{\partial x^4} - \rho I\left(1 + \frac{E}{KG}\right)\left(1 + \eta\frac{\partial \psi}{\partial t}\right)\frac{\partial^4 \psi}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{KG}\frac{\partial^4 \psi}{\partial t^4} + \rho A\left(1 + \eta\frac{\partial \psi}{\partial t}\right)\frac{\partial^2 \psi}{\partial t^2} = 0. \tag{6}$$

Equations (5) and (6) are the governing differential equations for transverse vibrations of viscoelastic Timoshenko beams. Furthermore, elastic Timoshenko results can be obtained for $\eta = 0$.

2.2 Modal analysis

Assume that the beam is excited harmonically with a frequency f (Soares *et al.*, 2016):

$$v(\xi, t) = V(\xi)\, e^{jft}, \quad \psi(\xi, t) = \Psi(\xi)\, e^{jft}, \quad \text{and} \quad \xi = \frac{x}{L}, \tag{7}$$

where $j = \sqrt{-1}$, ξ is the non-dimensional length of the beam, f is the natural frequency and $V(\xi)$ and $\Psi(\xi)$ are the normal functions of $v(x)$ and $\psi(x)$ respectively. Substituting the relations presented in Eq. (7) through Eqs. (5 - 6) and omitting the common term e^{jft} we obtain:

$$\left(1 + \eta\frac{\partial V}{\partial t}\right)^2 \frac{\partial^4 V}{\partial \xi^4} - b^2\left(r^2 + s^2\right)\left(1 + \eta\frac{\partial V}{\partial t}\right)\frac{\partial^4 V}{\partial \xi^2 \partial t^2} + b^4 r^2 s^2 \frac{\partial^4 V}{\partial t^4} + b^2\left(1 + \eta\frac{\partial V}{\partial t}\right)\frac{\partial^2 V}{\partial t^2} = 0, \tag{8}$$

$$\left(1 + \eta\frac{\partial \Psi}{\partial t}\right)^2 \frac{\partial^4 \Psi}{\partial \xi^4} - b^2\left(r^2 + s^2\right)\left(1 + \eta\frac{\partial \Psi}{\partial t}\right)\frac{\partial^4 \Psi}{\partial \xi^2 \partial t^2} + b^4 r^2 s^2 \frac{\partial^4 \Psi}{\partial t^4} + b^2\left(1 + \eta\frac{\partial \Psi}{\partial t}\right)\frac{\partial^2 \Psi}{\partial t^2} = 0. \tag{9}$$

where b, r and s are dimensionless parameters related with the effects of bending, rotation and shear deformation given by:

$$b^2 = \frac{\rho A L^4}{EI}, \quad r^2 = \frac{I}{AL^2} \quad \text{and} \quad s^2 = \frac{EI}{KGAL^2}. \tag{10}$$

We must consider two cases when obtaining Timoshenko beam model spatial solution. In the first case, assume (Azevedo *et al.*, 2016):

$$\sqrt{(r^2 - s^2)^2 + 4/b_v^2} > (r^2 + s^2) \quad \text{which leads to} \quad b_v < \frac{1}{r\,s} \tag{11}$$

while in the second

$$\sqrt{(r^2 - s^2)^2 + 4/b_v^2} < (r^2 + s^2) \quad \text{which leads to} \quad b_v > \frac{1}{r\,s}, \tag{12}$$

where

$$b_v = \frac{b\,f^2}{(1 + \eta j f)}. \tag{13}$$

We call this cutoff value $b_{crit} = 1/(r\,s)$. When $b_v < b_{crit}$ the solutions of Eqs. (5) and (6) can be expressed respectively, in trigonometric and hyperbolic functions as follows:

$$V(\xi) = C_1 cosh(\alpha_1\,\xi) + C_2 sinh(\alpha_1\,\xi) + C_3 cos(\beta\,\xi) + C_4 sin(\beta\,\xi), \tag{14}$$

$$\Psi(\xi) = C_1' sinh(\alpha_1\,\xi) + C_2' cosh(\alpha_1\,\xi) + C_3' sin(\beta\,\xi) + C_4' cos(\beta\,\xi), \tag{15}$$

with

$$\alpha_1 = \frac{b_v\sqrt{-\left(r^2 + s^2\right) + \sqrt{\left(\left(r^2 - s^2\right)^2 + 4/b_v^2\right)}}}{\sqrt{2}} \quad \text{and} \quad \beta = \frac{b_v\sqrt{\left(r^2 + s^2\right) + \sqrt{\left(\left(r^2 - s^2\right)^2 + 4/b_v^2\right)}}}{\sqrt{2}}. \tag{16}$$

When $b_v > b_{crit}$ the solutions $V(\xi)$ and $\Psi(\xi)$ can be expressed only in trigonometric functions:

$$V(\xi) = \overline{C}_1 cos(\alpha_2\,\xi) + \overline{C}_2 sin(\alpha_2\,\xi) + \overline{C}_3 cos(\beta\,\xi) + \overline{C}_4 sin(\beta\,\xi), \tag{17}$$

$$\Psi(\xi) = \overline{C}_1' sin(\alpha_2\,\xi) + \overline{C}_2' cos(\alpha_2\,\xi) + \overline{C}_3' sin(\beta\,\xi) + \overline{C}_4' cos(\beta\,\xi), \tag{18}$$

with

$$\alpha_2 = \frac{b_v\sqrt{\left(r^2 + s^2\right) - \sqrt{\left(\left(r^2 - s^2\right)^2 + 4/b_v^2\right)}}}{\sqrt{2}}. \tag{19}$$

The relations between the coefficients in Eqs. (14 - 15), or Eqs. (17 - 18) can be found in Huang (1961). Apllication of boundary condition will result in a infinite series of natural frequencies f_n, each associated with a particular mode shape. Huang and Huang (1971) observed that boundary conditions are the same for both elastic ($\eta = 0$) and viscoelastic beams, some of these conditions are presented in Tab. 1.

Table 1. Boundary conditions.

Boundary Condition	Shear Force	Moment	Total Slope	Deflection
Hinged	-	$\partial\Psi(\xi)/\partial\xi = 0$	-	$V(\xi) = 0$
Clamped	-	-	$\Psi(\xi) = 0$	$V(\xi) = 0$
Free	$\Psi(\xi) - \frac{1}{L}\partial V(\xi)/\partial\xi = 0$	$\partial\Psi(\xi)/\partial\xi = 0$	-	-
Sliding	$\Psi(\xi) - \frac{1}{L}\partial V(\xi)/\partial\xi = 0$	-	$\Psi(\xi) = 0$	-

2.3 Wave propagation

Consider an infinite length beam as a waveguide of flexural waves. The general solution for propagation of waves can be assumed as (Wang and So, 2005):

$$v(\xi,t) = D(\xi)\, e^{j(\kappa\xi - wt)} \quad \text{and} \quad \psi(\xi,t) = D'(\xi)\, e^{j(\kappa\xi - wt)}, \tag{20}$$

where $D(\xi)$ and $D'(\xi)$ are amplitudes, κ, the wavenumber, and w, the frequency. Substituting Eq. (20) through Eq. (5) or Eq. (6), and omitting the common terms, we obtain the following characteristic equation:

$$\kappa^4(1+j\,\eta\,w)^2 - \kappa^2 b^2 w^2 (r^2+s^2)(1+j\,\eta\,w) + b^4 r^2 s^2 w^4 - b^2 w^2 (1+j\,\eta\,w) = 0, \tag{21}$$

or

$$\kappa^4 - \kappa^2 b^2 (r^2+s^2)\, Z + b^4 r^2 s^2 Z^2 - b^2 Z = 0, \tag{22}$$

with

$$Z = \frac{w^2}{1+j\,\eta\,w}. \tag{23}$$

Solving Eq. (22) for Z, two real and positive roots are obtained:

$$Z_{1,2} = \frac{\kappa^2\left(1/\kappa^2 + r^2 + s^2 \pm \sqrt{\left(1/\kappa^2 + r^2 + s^2\right)^2 - 4\,r^2 s^2}\right)}{2\,b^2 r^2 s^2}. \tag{24}$$

After this procedure, the dispersion relationship of viscoelastic beam can be written as:

$$w_{1,2} = \frac{\eta j Z_{1,2} + \sqrt{4Z_{1,2} - \eta^2 Z_{1,2}^2}}{2} \quad \text{and} \quad w_{3,4} = \frac{\eta j Z_{1,2} - \sqrt{4Z_{1,2} - \eta^2 Z_{1,2}^2}}{2}. \tag{25}$$

Observe that the pairs of complex frequencies obtained ($\omega_{1,2}$ and $\omega_{3,4}$) differs on sign of real parts. Some authors usually denotes $\omega_{1,2}$ as first and second spectrum of Timoshenko beam (Bhaskar, 2009). These two branches of frequencies are related with dilatation and distortion elastic waves, that propagates when a solid medium is deformed (Kolsky, 1964; Soares and Hoefel, 2016). Also, frequencies real value vanishes for:

$$Z_{1,2} \geq \frac{4}{\eta^2}, \tag{26}$$

Substituting Eq. (24) into Eq. (26), we obtain two cut-off values for wavenumber κ:

$$\kappa_{1,2} = \frac{\sqrt{2}}{\eta}\sqrt{\frac{-b\,\left(\eta^2 - 4b^2 r^2 s^2\right)\left(b\,(r^2+s^2) \pm \sqrt{\eta^2 + (br^2 - bs^2)^2}\right)}{\eta^2 + 2\,b^2\,(r^4+s^4) \pm 2\,b\,(r^2+s^2)\sqrt{\eta^2 + (br^2 - bs^2)^2}}}, \tag{27}$$

these coefficients are the wavenumbers values where frequencies real parts turns to zero. Notice that for elastic beam, Eq. (24) turns to:

$$w_{1,2} = \left[\frac{\kappa^2\left(1/\kappa^2 + r^2 + s^2 \pm \sqrt{\left(1/\kappa^2 + r^2 + s^2\right)^2 - 4\,r^2 s^2}\right)}{2\,b^2 r^2 s^2}\right]^{1/2}. \tag{28}$$

which frequencies obtained are always real values, differently of viscoelastic material behaviour.

Usually, the dispersion relations of flexural waves are discussed in terms of phase speed c and group velocity c_g, written as (Rayleigh, 1877):

$$c = \frac{w}{\kappa} \quad \text{and} \quad c_g = \frac{\partial w}{\partial \kappa}. \tag{29}$$

Phase speed is defined as the the speed of the individual particles that propagates in the structure, and is not associated with transfer of any physical quantity. Group velocity is associated with the propagation of a group of waves of similar frequencies, travelling in bundles.

3. RESULTS AND DISCUSSION

This section presents the dispersion relations results and phase speed obtained for an elastic and viscoelastic Timoshenko beam (Figs. 2 - 5), based on literature example (Manevich and Kolakowski, 2011). Further, first five natural frequencies are calculated for different values of viscosity parameters to hinged-hinged and clamped-clamped boundary conditions, as shown on Tabs. 3 and 4.

3.1 Dispersion relations

Figures 2 and 3 presents frequency of waves propagation in function of wavenumbers for various coefficients of viscosity. Continuous curves represents elastic beam results, dashed lines are for $\eta = 0.5\, Pa \cdot s$, and dot dashed curves are for $\eta = 1.0\, Pa \cdot s$. Finally, reference results are marked by "o".

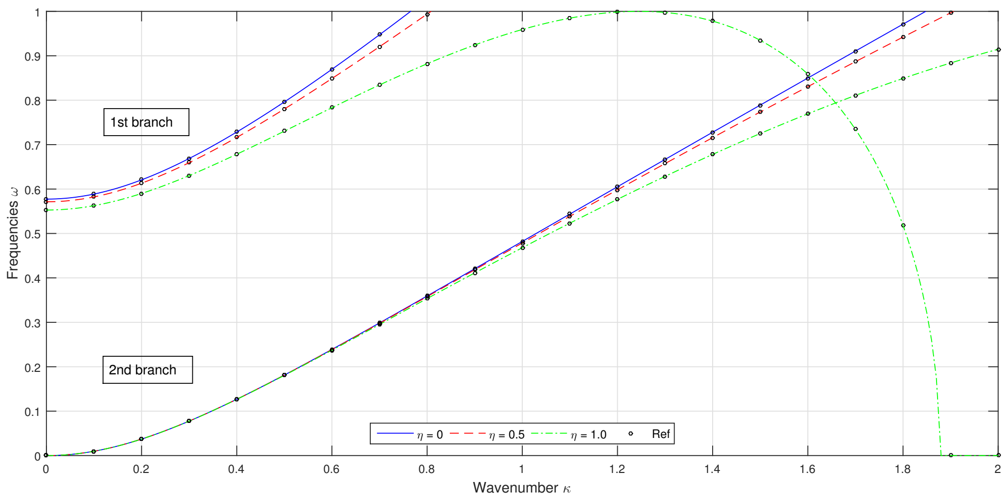

Figure 2. Frequencies of waves propagation for first wavenumbers range

Bhaskar (2009) observed that the lower branch is an improvement on the EBT, provided by the addiction of rotatory inertia and shear deflection, while the higher branch is a result of an extra flexural degree of freedom granted to cross section on Timoshenko model. Notice that frequencies obtained for viscoelastic material becomes lower than elastic results, as wavenumber or viscosity coefficient increases. Manevich and Kolakowski (2011) noted that in distinction to the elastic material, frequencies becomes to decrease at large κ, vanishing if wavenumber is high enough. Furthermore, the cut-off wavenumbers for $\eta = 1.0$ obtained are $\kappa_1 = 1.87826$ and $\kappa_2 = 3.93797$, while for $\eta = 0.5$ were $\kappa_1 = 3.53159$ and $\kappa_2 = 6.96365$.

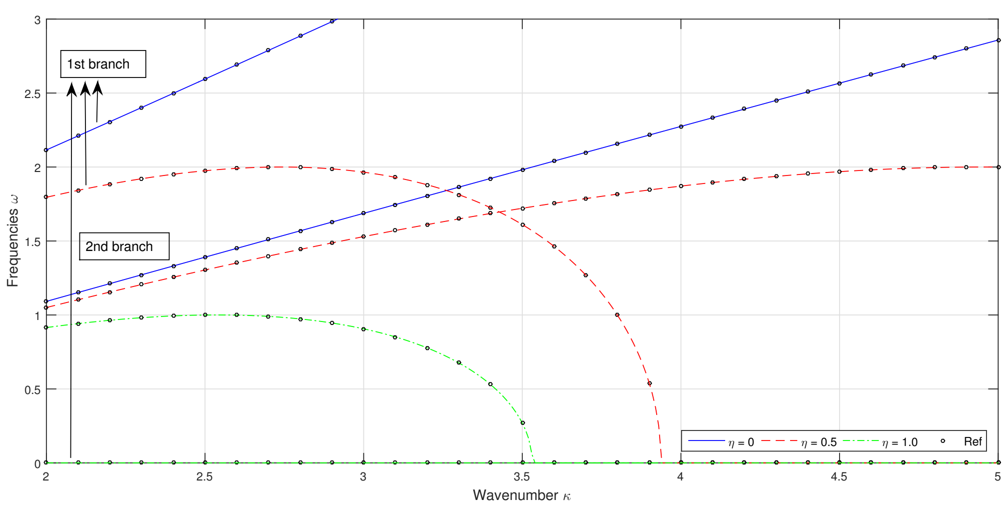

Figure 3. Frequencies of waves propagation for second wavenumbers range

Similarly to observed on first frequencies branch, real parts on second frequencies branch also vanishes for high enough wavenumbers. Complex frequencies and purely imaginary frequencies effects on viscoelastic material will be showed on phase speed plots.

Figures 4 and 5 presents phase speed for same parameters and wavenumbers range specified for Figs. 2 and 3.

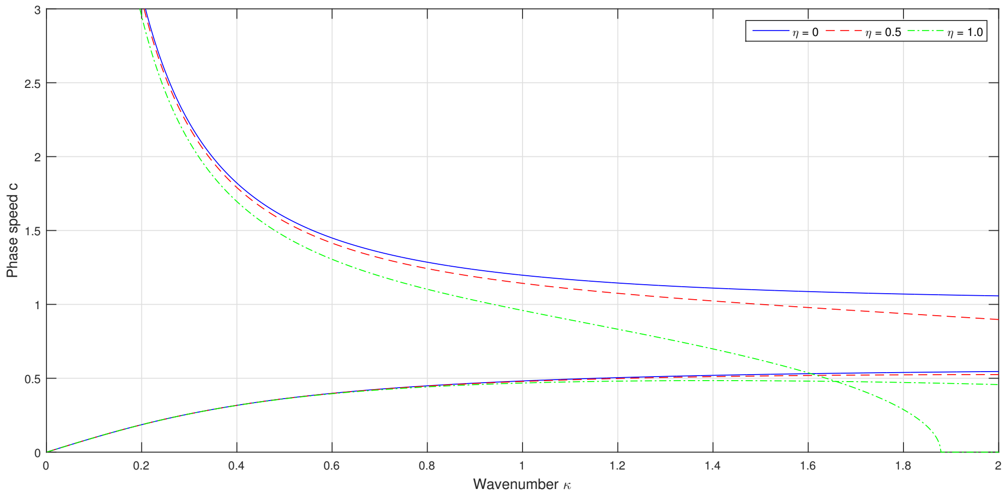

Figure 4. Phase speed for first wavenumbers range

Observe that dispersive waves nature is confirmed on phase speed plots for both branchs of waves propagated, as the velocities vary in function of wavenumber κ. Also, viscoelastic beam results showns that for complex frequencies values phase speed is attenuated as waves propagate, and when the frequency is purelly imaginary the system will not allow any way to propagation (Gopalakrishnan and Narendar, 2013).

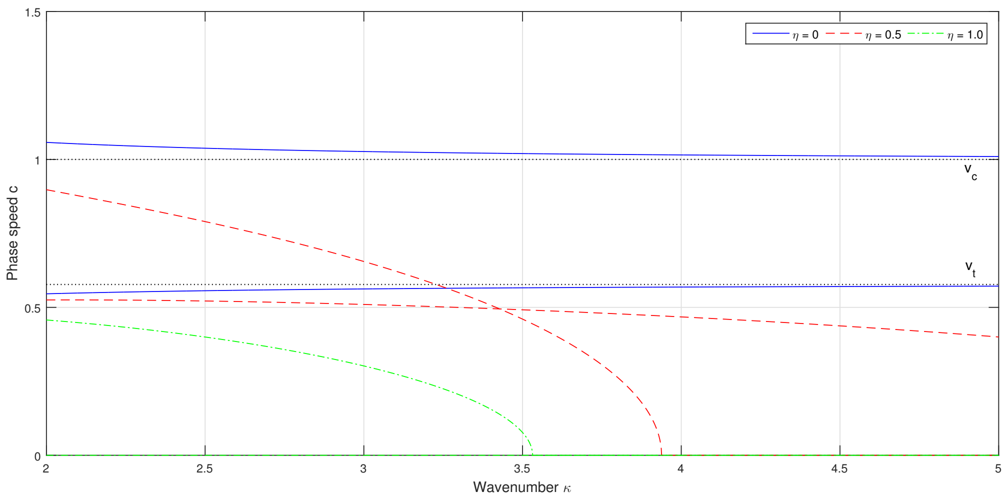

Figure 5. Phase speed for second wavenumbers range

Figure 5 presents the elastic and viscoelastic phase speeds for the second range analysed. Smith (2008) discuss that on elastic material, first branch of results approaches the compressional waves speed $v_c = \sqrt{E/\rho}$ in the limit of high wavenumbers, while the second branch approaches the transverse wave velocity $v_t = \sqrt{KG/\rho}$, as show on dotted lines. In literature's example these velocity parameters are $v_c = 1$ and $v_t = 0.57735\, m/s$.

3.2 Hinged-hinged beam

Consider a finite length hinged-hinged beam, shown in Fig. 6, whose properties are given in Tab. 2. In this section, viscoelastic beams are calculated for two viscosity parameters: $\eta_1 = 1 \times 10^{-8}\, Pa \cdot s$ and $\eta_2 = 2 \times 10^{-8}\, Pa \cdot s$. The same parameters values are considered for clamped-clamped boundary conditions.

Table 2. Properties of the beam.

$L\,[m]$	$A\,[m^2]$	$I\,[m^4]$	$K\,[\;]$	$E\,[Pa]$	$G\,[Pa]$	$\rho\,[Kg/m^3]$
0.5	1.5625×10^{-2}	2.0345×10^{-5}	5/6	210×10^9	80.8×10^{-5}	7850

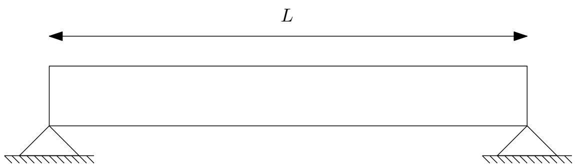

Figure 6. Hinged-hinged beam

Table 3 presents the first five natural frequencies f_n calculated for elastic and viscoelastics beams.

Table 3. First four natural frequencies of a hinged-hinged beam for various η parameters.

Hinged-hinged	$f_n(\eta_o = 0)$	$f_n(\eta_1 = 1 \times 10^{-8})$	$f_n(\eta_2 = 2 \times 10^{-8})$
First mode	6712.45	6712.43+0.22 j	6712.36+0.45 j
Second mode	22137.07	22031.98+2.45 j	21713.65+4.90 j
Third mode	40705.44	32407.97+8.28 j	27759.92 j
Fourth mode	60176.72	10110.84 j	227502.07 j
Fifth mode	79816.46	35528.48 j	723894.29 j

Notice that natural frequencies appear in the complex form on viscoelastic results. Although frequencies real part for first and second modes were close to elastic results, the difference arises as the mode of vibration or viscous parameter increases. Also, Tab. 3 shows that real part of frequencies vanishes for modes of vibration beyond third mode on η_1 results, and beyond second mode on η_2. Ozhan and Pakdemirli (2013) discussed that on viscoelastic beams, frequencies real parts can be considered as natural frequencies while imaginary parts may be seen as damping effects. Therefore, is expected that viscoelastic material attenuate oscillations of the beam.

3.3 Clamped-clamped beam

Consider now a clamped-clamped beam, shown in Fig. 7. Natural frequencies obtained for elastic and viscoelastic beams were disposed on Tab. 4.

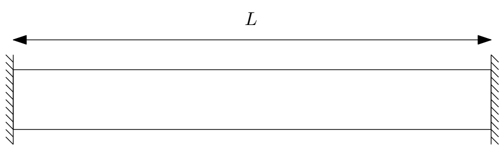

Figure 7. Clamped-clamped beam

Table 4. First four natural frequencies of a clamped-clamped beam for various η parameters.

Clamped-clamped	$f_n(\eta = 0)$	$f_n(\eta = 1 \times 10^{-8})$	$f_n(\eta = 2 \times 10^{-8})$
First mode	12286.71	12285.01+0.76 j	12279.91+1.51 j
Second mode	26911.11	26496.57+3.62 j	25212.07+7.42 j
Third mode	43916.54	28548.45+9.64 j	50277.92 j
Fourth mode	61800.21	115371.69 j	254366.10 j
Fifth mode	80366.62	365553.78 j	744242.12 j

Table 4 shows that clamped-clamped boundary conditions have natural frequencies higher than hinged-hinged beam, and the presence of complex and pure imaginary frequencies for beams with internal friction is confirmed. Note that complex part of frequencies increases as viscous parameter becomes higher, and that real part of frequencies vanishes similarly to discussed on hinged-hinged example. Can-zhang *et al.* (1990) observed that differences between elastic and viscoelastic beams would be small under low frequency, low material viscosity and long beam conditions.

4. CONCLUSIONS

This paper discusses the effects of internal friction on TBT frequencies and dispersion relations. It was observed that differently of expected for elastic materials, natural frequencies on viscoelastic beams are complex or pure imaginary. Also, frenquencies real part decreases with the increase of viscosity parameter, while imaginary part become higher. Futhermore, the attenuation behaviour of viscoelastic material was discussed in the light of waves propagation approach, and the implications of complex and pure imaginary frequencies on phase speeds. Finally, it was shown on numerical examples that both frequencies branchs real part vanishes for sufficiently high wavenumbers.

5. REFERENCES

Azevedo, C., Soares, A. and Hoefel, S., 2016. “Finite element analysis of shear-deformation and rotatory inertia for beam vibration.” *Revista Interdisciplinar de Pesquisa em Engenharia*, Vol. 2 (34), pp. 86–103.

Bhaskar, A., 2009. “Elastic waves in timoshenko beams: the ’lost and found’ of an eigenmode”. *Proceedings of the Royal Society of London Series A*, Vol. 465, pp. 239–255.

Can-zhang, X., Yi-zhou, J. and Bao-ping, C., 1990. “General dynamic equation and dynamical characteristics of viscoelastic timoshenko beams”. *Applied Mathematics and Mechanics*, Vol. 11 (2), pp. 177–184.

Chen, L.Q., Peng, L., Zhang, A.Q. and Ding, H., 2017. “Transverse vibration of viscoelastic timoshenko beam-columns”. *Journal of Vibration and Control*, Vol. 23 (10), pp. 1572–1584.

Gopalakrishnan, S. and Narendar, S., 2013. *Wave Propagation in Nanostructures: Nonlocal Continuum Mechanics Formulations*. NanoScience and Technology. Springer International Publishing.

Huang, T.C., 1961. “The effect of rotatory inertia and of shear deformation on the frequency and normal mode equations of uniform beams with simple end conditions”. *Journal of Applied Mechanics*, Vol. 28, pp. 579–584.

Huang, T.C. and Huang, C.C., 1971. “Free vibrations of viscoelastic timoshenko beams”. *Journal of Applied Mechanics*, Vol. 38 (2), pp. 515–521.

Kolsky, H., 1964. “Stress waves in solids”. *Journal of Sound and Vibration*, Vol. 1, pp. 88–110.

Lang, H. and Leyendecker, S., 2016. “Complex frequency response for linear beams with Kelvin-Voigt viscoelastic material”. In *4th Joint International Conference on Multibody System Dynamics*.

Mandan, V.P., 1969. “On longitudinal vibrations of a semi-infinite voigt-kelvin rod with non-uniform viscoelastic properties”. *Journal of Mathematical Analysis and Applications*, Vol. 28, pp. 88–92.

Manevich, A. and Kolakowski, Z., 2011. “Free and forced oscillations of timoshenko beam made os viscoelastic material”. *Journal of Mathematical Analysis and Applications*, Vol. 49 (1), pp. 3–16.

Martin, O., 2014. “Quasi static and dynamic analysis for viscoelastic beams with the constitutive equation in a hereditary integral form”. Vol. 5 (LXIII), pp. 329–341.

Marynowski, K., 2002. “Non-linear dynamic analysis of an axialy moving viscoelastic beam”. *Journal of Theoretical and Applied Mechanics*, Vol. 2 (40), pp. 465–482.

Ozhan, B.B. and Pakdemirli, M., 2013. “Effect of viscoelasticity on the natural frequencies of axially moving continua”. *Advances in Mechanical Engineering*, Vol. 5, p. 169598.

Rayleigh, 1877. “On progressive waves”. In *Proceedings of the London Mathematical Society*. London, Vol. IX, pp. 21–26.

Smith, R.W.M., 2008. “Graphical representation of timoshenko beam modes for clampedâĂŞclamped boundary conditions at high frequency: Beyond transverse deflection”. *Wave Motion*, Vol. 45, pp. 785–794.

Soares, A., Azevedo, A.C. and Hoefel, S., 2016. “Dynamic analysis of elastically supported timoshenko beam.” *Revista Interdisciplinar de Pesquisa em Engenharia*, Vol. 2 (34), pp. 71–85.

Soares, A. and Hoefel, S., 2016. “Analysis of rotatory inertia and shear deformation on transverse vibration of beams.” In *Proceedings of the IX Congresso Nacional de Engenharia Mecânica - CONEM 2016*. Fortaleza, Brasil.

Timoshenko, S.P., 1921. “On the correction for shear of the differential equation for transverse vibration of prismatic bars”. *Philosophical Magazine*, Vol. 41, pp. 744–746.

Wang, X.Q. and So, R.M.C., 2005. “Various standing waves in a timoshenko beam”. *Journal of Sound and Vibration*, Vol. 280, pp. 311–328.

6. RESPONSIBILITY NOTICE

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