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# HIGH-ORDER IMMERSED INTERFACE METHOD FOR 2D POISSON EQUATION

**Miguel Angel Rojas Meza**
**Leandro Franco de Souza**
**José Alberto Cuminato**
Department of Applied Mathematics and Statistics, University of São Paulo, Av. Trabalhador São Carlense, 400, São Carlos, Brazil
migrojas@usp, lefraso@icmc.usp.br, jacumina@icmc.usp.br

***Abstract.****The main objective of the present work is to show high-order numerical procedures that are able to handle discontinuities in Poisson Equation. The method used is a Immersed Interface Method, adopting jump correction terms in the finite difference approximations. The results show that 2nd, 4th and 8th order results were obtained when the jump corrections are added in the solution.*

***Keywords:*** *Immersed Interface Method, High Order approximations, Poisson Equation*

## 1. INTRODUCTION

In numerical studies when it is necessary to simulate a physical phenomena that contains discontinuities, it is required to implement numerical schemes that can handle these discontinuities. In multiphase flows a Poisson equation is used to obtain the pressure field. The discontinuity along fluids with different proprieties requires numerical procedures that decreases the overall order of convergence in these regions. The equations can be solved in a conform mesh, but when the physical phenomena are not stationary, a great effort is carried out to move the mesh at each time step. A way to overcome this problem is the use of immersed boundary methods, with the drawback of lowering the accuracy order in the interface.

The method known as Immersed Interface Method (IIM) was developed by Leveque and Li (1994) to improve the accuracy order of the Immersed Border Method that was initially of order one. This method treats equations with discontinuous coefficients and are used to solve numerically initial value and contour problems in domains with irregular geometries. To model the discontinuities in the interface, the coefficients in the finite differences are modified and terms of correction are added and determined depending on the jump function as it is proposed in Linnick and Fasel (2005).

In this work we explain the Immersed Interface Method, detailing finite difference explicit schemes of second, fourth, and high order of convergence accuracy, which are used in the discretization of the Poisson equation. In the last section we show some results verifying the convergence orders of the different methods.

## 2. COMPUTACIONAL PROCEDURE

Consider a function $f(x)$ with a discontinuity at the point $x = x_\alpha$. We want to use the Taylor series expansion at point $x_i$ to approximate $f(x)$ at point $x_{i+1}$. Assume that $f(x)$ is analytic in all the domain points $D = \{x|x_{i-1} \leq x \leq x_{i+1}\}$ except at $x_\alpha$, where there is a jump (discontinuity) in the value of the function and/or its derivatives. If $x_i < x_\alpha$ the standard Taylor series expansion involving $x_\alpha$ can not be used to approximate $f(x_{i+1})$ unless a term correction $J_\alpha$ is added, namely

$$f(x_{i+1}) = f(x_i) + f^{(1)}(x_i)h + f^{(2)}(x_i)\frac{h^2}{2!} + \cdots + J_\alpha, \quad (1)$$

where

$$J_\alpha = [f]_\alpha + [f^{(1)}]_\alpha(h^+) + \frac{1}{2!}[f^{(2)}]_\alpha(h^+)^2 + \cdots,$$

with $h = x_{i+1} - x_i$ e $h^+ = x_{i+1} - x_\alpha$. The notation $[\phi]_\alpha$ represents the jump in the value of a function $\phi$ at the point $x = x_\alpha$, given by

$$[\phi]_\alpha = \lim_{x\to x_\alpha^+} \phi(x) - \lim_{x\to x_\alpha^-} \phi(x),$$

The equation (1) is called Taylor series corrected and its existence for the function $f$ is proven by Wiegmann and Bube (2000). Using the correction term $J_\alpha$ any finite difference method can be modified in such a way that the corrected method will maintain the order of accuracy of the original this when stencil involves a singularity.

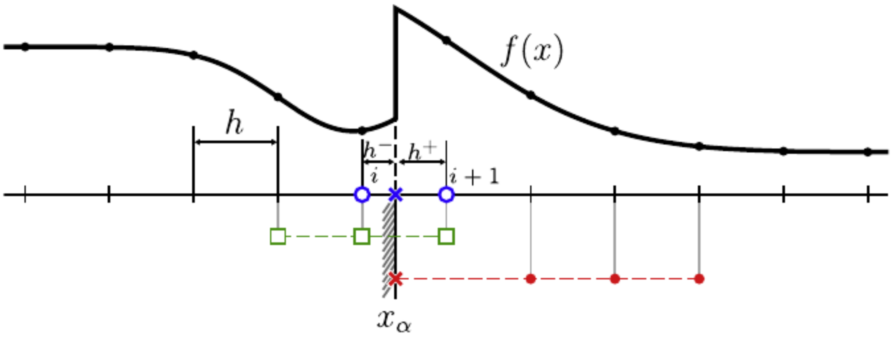



Figure 1. Example of explicit domain discretization for the second-order method.

### 2.1 Explicit discretization scheme

The coefficients of the different finite difference schemes to represent the second derivative were obtained from Fornberg (1988). In this article are presented several tables with different stencils that change depending on the position of the points where a differential equation is generated.To obtain second order convergence in the second spatial derivative, we use the numerical scheme

$$f_i^2 = R_{i-1}^2 f_{i-1} + R_i^2 f_i + R_{i+1}^2 f_{i+1} + C_I, \tag{2}$$

where $R_{i-1}^2 = R_{i+1}^2 = \frac{1}{h^2}$, $R_i^2 = -\frac{2}{h^2}$ e $C_I = -R_I^2 J_{\alpha 0}$.

In this scheme, $I = i+1$ if the jump occurs in $x_i < x_\alpha < x_{i+1}$, and in this case $h^+ = x_{i+1} - x_\alpha$ and

$$J_{\alpha 0} = [f^{(0)}]_\alpha + (h^+)[f^{(1)}]_\alpha + \frac{(h^+)^2}{2!}[f^{(2)}]_\alpha + \frac{(h^+)^3}{3!}[f^{(3)}]_\alpha. \tag{3}$$

If the jump occurs in $x_{i-1} < x_\alpha < x_i$, then $I = i-1$ and in this case $h^- = x_\alpha - x_{i-1}$ e

$$J_{\alpha 0} = -[f^{(0)}]_\alpha + (h^-)[f^{(1)}]_\alpha - \frac{(h^-)^2}{2!}[f^{(2)}]_\alpha + \frac{(h^-)^3}{3!}[f^{(3)}]_\alpha. \tag{4}$$

The jumps can be obtained by

$$[f^{(n)}]_\alpha = f_+^{(n)} - f_-^{(n)}, \tag{5}$$

where $f_+^{(n)}$ e $f_-^{(n)}$ can be obtained by interpolations

$$f_+^{(n)} = c_{n_{\alpha+}} f_\alpha + c_{n_{i+2}} f_{i+2} + c_{n_{i+3}} f_{i+3} + c_{n_{i+4}} f_{i+4},$$

$$f_-^{(n)} = c_{n_{\alpha-}} f_\alpha + c_{n_{i-1}} f_{i-1} + c_{n_{i-2}} f_{i-2} + c_{n_{i-3}} f_{i-3}.$$

The coefficients $c_n$ to calculate

$$f_\alpha^{(n)} = c_\alpha f_\alpha + c_i f_i + c_{i+1} f_{i+1} + c_{i+2} f_{i+2}, \tag{6}$$

are obtained from the linear system resolution

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & h_i & h_{i+1} & h_{i+2} \\ 0 & h_i^2 & h_{i+1}^2 & h_{i+2}^2 \\ 0 & h_i^3 & h_{i+1}^3 & h_{i+2}^3 \end{bmatrix} \begin{bmatrix} c_\alpha \\ c_i \\ c_{i+1} \\ c_{i+2} \end{bmatrix} = \begin{bmatrix} 0!\delta_{n0} \\ 1!\delta_{n1} \\ 2!\delta_{n2} \\ 3!\delta_{n3} \end{bmatrix}, \tag{7}$$

where $h_i = x_i - x_\alpha$ and $\delta_{ij}$ is the Kronecker delta function.

An explicit finite difference scheme a fourth-order, five-point and one-dimensional model will be used to approximate numerically the second spatial derivative, the scheme is given by

$$f_i^{(2)} = -\frac{1}{12h^2}f_{i-2} + \frac{4}{3h^2}f_{i-1} - \frac{5}{2h^2}f_i + \frac{4}{3h^2}f_{i+1} - \frac{1}{12h^2}f_{i+2} + C_I, \tag{8}$$

with $C_I = -(R_I^1 J_{\alpha 0}^* + R_I^2 J_{\alpha 0})$, on what

$$\begin{cases} R_I^1 = 0 \quad \text{e} \quad R_I^2 = \frac{4}{3h^2} \quad \text{se} \quad I = i-1 \quad ou \quad I = i+2, \\ R_I^1 = -\frac{1}{12h^2} \quad \text{e} \quad R_I^2 = \frac{4}{3h^2} \quad \text{se} \quad I = i \quad ou \quad I = i+1, \end{cases} \tag{9}$$

In these schemes the jump occurs in $x_i < x_\alpha < x_{i+1}$ so, when $I = i+1$ or $I = i+2$, in these cases $h^+ = x_{i+1} - x_\alpha$,

$$J_{\alpha 0} = \sum_{k=0}^{5} \frac{(h^+)^k}{k!}[f^{(k)}]_\alpha, \qquad J_{\alpha 0}^* = \sum_{k=0}^{5} \frac{(h+h^+)^k}{k!}[f^{(k)}]_\alpha,$$

where $I = i$ ou $I = i-1$, in these cases $h^- = x_\alpha - x_{i-1}$ and

$$J_{\alpha 0} = \sum_{k=0}^{5} (-1)^{k+1}\frac{(h^-)^k}{k!}[f^{(k)}]_\alpha, \qquad J_{\alpha 0}^* = \sum_{k=0}^{5} (-1)^{k+1}\frac{(h+h^-)^k}{k!}[f^{(k)}]_\alpha,$$

where

$$f_\alpha^{(n)} = c_\alpha f_\alpha + c_i f_i + c_{i+1} f_{i+1} + c_{i+2} f_{i+2} + c_{i+3} f_{i+3} + c_{i+4} f_{i+4}, \tag{10}$$

and

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & h_i & h_{i+1} & h_{i+2} & h_{i+3} & h_{i+4} \\ 0 & h_i^2 & h_{i+1}^2 & h_{i+2}^2 & h_{i+3}^2 & h_{i+4}^2 \\ 0 & h_i^3 & h_{i+1}^3 & h_{i+2}^3 & h_{i+3}^3 & h_{i+4}^3 \\ 0 & h_i^4 & h_{i+1}^4 & h_{i+2}^4 & h_{i+3}^4 & h_{i+4}^4 \\ 0 & h_i^5 & h_{i+1}^5 & h_{i+2}^5 & h_{i+3}^5 & h_{i+4}^5 \end{bmatrix} \begin{bmatrix} c_\alpha \\ c_i \\ c_{i+1} \\ c_{i+2} \\ c_{i+3} \\ c_{i+4} \end{bmatrix} = \begin{bmatrix} 1\delta_{n0} \\ 1\delta_{n1} \\ 2!\delta_{n2} \\ 3!\delta_{n3} \\ 4!\delta_{n4} \\ 5!\delta_{n5} \end{bmatrix}, \tag{11}$$

the interpolations to represent the terms $[f^{(n)}]_\alpha$ can be calculated by the formulas 10, where the coefficients of each linear combination is obtained from the system of Eq. (11).

To obtain a sixth order scheme we will consider the following centered stencil of seven points

$$f_i^{(2)} = \frac{1}{90h^2}f_{i-3} - \frac{3}{20h^2}f_{i-2} + \frac{3}{2h^2}f_{i-1} - \frac{49}{18h^2}f_i + \frac{3}{2h^2}f_{i+1} - \frac{3}{20h^2}f_{i+2} + \frac{1}{90h^2}f_{i+3} + C_I, \tag{12}$$

with $C_I = -(R_I^1 J_{\alpha 0}^* + R_I^2 J_{\alpha 0} + R_I^3 J_{\alpha 0}^{**})$, where

$$\begin{cases} R_I^1 = -\frac{3}{20h^2} \quad R_I^2 = \frac{3}{2h^2} \quad \text{e} \quad R_I^3 = \frac{1}{90h^2} \quad \text{se} \quad I = i \quad ou \quad I = i+1, \\ R_I^1 = \frac{1}{90h^2} \quad R_I^2 = -\frac{3}{20h^2} \quad \text{e} \quad R_I^3 = 0 \quad \text{se} \quad I = i-1 \quad ou \quad I = i+2, \\ R_I^1 = 0 \quad R_I^2 = \frac{1}{90h^2} \quad \text{e} \quad R_I^3 = 0 \quad \text{se} \quad I = i-2 \quad ou \quad I = i+3. \end{cases} \tag{13}$$

In these schemes, the jump occurs in $x_i < x_\alpha < x_{i+1}$, when $I = i+1$, $I = i+2$ or $I = i+3$. In these cases $h^+ = x_{i+1} - x_\alpha$ , we defined

$$J_{\alpha 0} = \sum_{k=0}^{7} \frac{(h^+)^k}{k!}[f^{(k)}]_\alpha, \qquad J_{\alpha 0}^* = \sum_{k=0}^{7} \frac{(h+h^+)^k}{k!}[f^{(k)}]_\alpha, \qquad J_{\alpha 0}^{**} = \sum_{k=0}^{7} \frac{(2h+h^+)^k}{k!}[f^{(k)}]_\alpha.$$

when $I = i$, $I = i-1$ or $I = i-2$, in these cases $h^- = x_\alpha - x_{i-1}$ we have

$$J_{\alpha 0} = \sum_{k=0}^{7} (-1)^{k+1}\frac{(h^-)^k}{k!}[f^{(k)}]_\alpha, \qquad J_{\alpha 0}^* = \sum_{k=0}^{7} (-1)^{k+1}\frac{(h+h^-)^k}{k!}[f^{(k)}]_\alpha, \qquad J_{\alpha 0}^{**} = \sum_{k=0}^{7} (-1)^{k+1}\frac{(2h+h^-)^k}{k!}[f^{(k)}]_\alpha.$$

Again the jumps can be obtained by

$$[f^{(n)}]_\alpha = f_+^{(n)} - f_-^{(n)}, \tag{14}$$

where $f_+^{(n)}$ and $f_-^{(n)}$ can be obtained by interpolations

$$f_+^{(n)} = c_{n_{\alpha+}} f_\alpha + c_{n_{i+2}} f_{i+2} + c_{n_{i+3}} f_{i+3} + c_{n_{i+4}} f_{i+4} + c_{n_{i+5}} f_{i+5} + c_{n_{i+6}} f_{i+6} + c_{n_{i+7}} f_{i+7} + c_{n_{i+8}} f_{i+8},$$

$$f_-^{(n)} = c_{n_{\alpha-}} f_\alpha + c_{n_{i-1}} f_{i-1} + c_{n_{i-2}} f_{i-2} + c_{n_{i-3}} f_{i-3} + c_{n_{i-4}} f_{i-4} + c_{n_{i-5}} f_{i-5} + c_{n_{i-6}} f_{i-6} + c_{n_{i-7}} f_{i-7}.$$

the new coefficients $c_n$ to calculate

$$f_\alpha^{(n)} = c_\alpha f_\alpha + c_i f_i + c_{i+1} f_{i+1} + c_{i+2} f_{i+2} + c_{i+3} f_{i+3} + c_{i+4} f_{i+4} + c_{i+5} f_{i+5} + c_{i+6} f_{i+6}, \tag{15}$$

are obtained from the resolution of the new linear system

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & h_i & h_{i+1} & h_{i+2} & h_{i+3} & h_{i+4} & h_{i+5} & h_{i+6} \\ 0 & h_i^2 & h_{i+1}^2 & h_{i+2}^2 & h_{i+3}^2 & h_{i+4}^2 & h_{i+5}^2 & h_{i+6}^2 \\ 0 & h_i^3 & h_{i+1}^3 & h_{i+2}^3 & h_{i+3}^3 & h_{i+4}^3 & h_{i+5}^3 & h_{i+6}^3 \\ 0 & h_i^4 & h_{i+1}^4 & h_{i+2}^4 & h_{i+3}^4 & h_{i+4}^4 & h_{i+5}^4 & h_{i+6}^4 \\ 0 & h_i^5 & h_{i+1}^5 & h_{i+2}^5 & h_{i+3}^5 & h_{i+4}^5 & h_{i+5}^5 & h_{i+6}^5 \\ 0 & h_i^6 & h_{i+1}^6 & h_{i+2}^6 & h_{i+3}^6 & h_{i+4}^6 & h_{i+5}^6 & h_{i+6}^6 \\ 0 & h_i^7 & h_{i+1}^7 & h_{i+2}^7 & h_{i+3}^7 & h_{i+4}^7 & h_{i+5}^7 & h_{i+6}^7 \end{bmatrix} \begin{bmatrix} c_\alpha \\ c_i \\ c_{i+1} \\ c_{i+2} \\ c_{i+3} \\ c_{i+4} \\ c_{i+5} \\ c_{i+6} \end{bmatrix} = \begin{bmatrix} 1\delta_{n0} \\ 1\delta_{n1} \\ 2!\delta_{n2} \\ 3!\delta_{n3} \\ 4!\delta_{n4} \\ 5!\delta_{n5} \\ 6!\delta_{n6} \\ 7!\delta_{n7} \end{bmatrix}, \tag{16}$$

For the discretization of the 2D Poisson equation by explicit finite differences in points the second, fourth, and sixth orders that have been defined previously. For the points that are close to the immersed interface, one must add correction, then the schemes are denoted as follows

$$f_{i,j}^{(2)} = R_{i,j} f_{i,j} + C_{i,j}, \tag{17}$$

where $R_{i,j}$ will be the coefficients of each of the 3 schemes applied to each of the two directions separately.

Depending on the position of the point, the stencil will be used in each coordinate $(i, j)$ and $C_{i,j}$ will be the correction term of each respective method in each of the two directions, of this form

$$C_{i,j} = J_{\alpha x I(j)} + J_{\alpha x J(i)}, \tag{18}$$

$$J_{\alpha x I} = R_I^{xx} J_{\alpha 0 x},$$

$$J_{\alpha x J} = R_I^{yy} J_{\alpha 0 y},$$

$R_I^{xx}$ e $J_{\alpha 0 x}$ correspond to $R_i^2$ and $J_{\alpha 0}$ in the $x$ direction. The same goes for the $y$ direction.

$$I = \begin{cases} i-1, & \text{se } x_{i-1} < x_\alpha < x_i, \\ i+1, & \text{se } x_i < x_\alpha < x_{i+1} \end{cases}$$

$$J = \begin{cases} j-1, & \text{se } x_{j-1} < x_\alpha < x_j, \\ j+1, & \text{se } x_j < x_\alpha < x_{j+1} \end{cases}$$

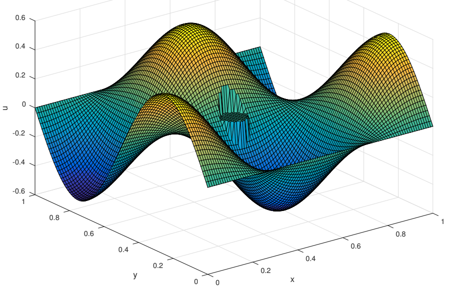
Note that $I = I(j)$ and $J = J(i)$, that is, the corrections are made separately for each direction using the one-dimensional procedure.

## 3. RESULTS

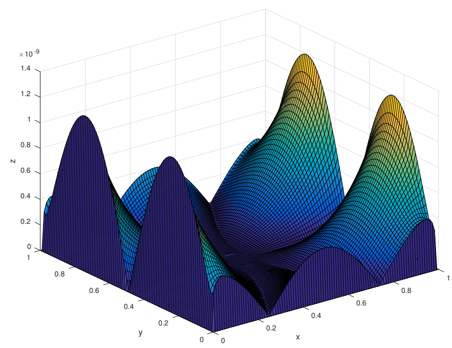
In order to verify the proposed method it is considered a 2D Poisson equation

$$\nabla \cdot (\beta \nabla u) = g \tag{19}$$

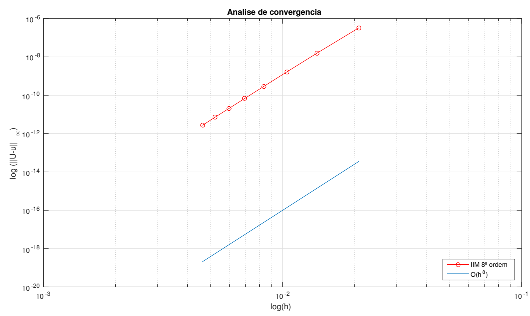
defined in a unit square, with value $\beta = 1$ where the term source $g$ and the exact solution $f$ are given by



(a) Numerical solution, mesh size $h = 0.01$



(b) Typical error distribution, $Error = 1.2067e-09$




(c) Syntactic error analysis on logarithmic scale

Figure 2. Eighth order methodology in the Poisson problem

$$g(x,y) = \begin{cases} 0, & \text{if } (x-0.5)^2 + (y-0.5)^2 \leq 0.05123^2 \\ -4\pi^2 \sin(2\pi x)\cos(2\pi y), & \text{otherwise,} \end{cases}$$

with solution $u(x, y) = 0.5 \sin(2\pi x) \cos(2\pi y)$, out of the circle. Figure 3(a) shows a typical result obtained in the presence of discontinuities. The discontinuity here is a cylinder were the values of the function and its derivatives are zero. In Fig. 3(b) the absolute error is plotted were it can be seen that the larger error are away from the discontinuity region. Table 1 shows the maximum error obtained in some simulations ($error_1$, $error_2$ and $error_3$), were 2nd , 4th and 6th order approximations were adopted and different mesh size were adopted. It can be seen that the convergence rate ($r1$, $r2$ and $r3$) are 2, 4 and around 8, respectively. This shows that the adopted immersed interface method is well implemented and the results are converging with the expected rate as shown in the Fig. 3(c).

Table 1. Error in calculating the approximate solution in the Poisson problem

| ***h*** | $\|U-u\|_\infty$ $error_1$ | ***r1*** | $\|U-u\|_\infty$ $error_2$ | ***r2*** | $\|U-u\|_\infty$ $error_3$ | ***r3*** |
|---|---|---|---|---|---|---|
| 1/72 | 3.2723e-04 | | 3.3281e-07 | | 3.2961e-07 | |
| 1/96 | 1.8418e-04 | 1.9978 | 1.0570e-07 | 3.9868 | 1.5491e-08 | 7.5411 |
| 1/120 | 1.1792e-04 | 1.9982 | 4.3164e-08 | 4.0136 | 1.6608e-09 | 7.7619 |
| 1/144 | 8.1898e-05 | 1.9996 | 2.0808e-08 | 4.0020 | 2.8841e-10 | 7.8455 |
| 1/168 | 6.0172e-05 | 1.9999 | 1.1226e-08 | 4.0036 | 6.8507e-11 | 7.8840 |
| 1/192 | 4.6063e-05 | 2.0009 | 6.5855e-09 | 3.9940 | 2.0276e-11 | 7.8983 |
| 1/216 | 3.6401e-057 | 1.9988 | 4.1092e-09 | 4.0043 | 7.0808e-12 | 7.8786 |
| 1/240 | 2.9486e-05 | 1.9994 | 2.6957e-09 | 4.0013 | 2.7524e-12 | 8.0226 |

## 4. CONCLUSIONS

In present work it is proposed some approximation schemes for IIM. 2nd, 4th and 6th order approximations were adopted. The results show that the IIM was able to simulate and solve the 2D Poisson equation with the presence of discontinuities without lowering the order of the results and obtaining a super convergence in the high order method.

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