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QUANTIFICATION OF MATERIAL UNCERTAINTIES IN MODAL ANALYSIS OF A PLATE USING A GAUSSIAN PROCESS AND FINITE ELEMENT METHOD

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Abstract. *Numerical simulations are very useful in designs to predict what response a system will have after it is built. Those predictions, however, are bounded to uncertainty due to many things as simplifications of the models, lack of knowledge of the exact real state of a system, materials properties deviations and so on. This work focus in the study of that last type of uncertainty in computational modal analysis, the goal is to see how to quantify, with the minimum computational cost possible, property variation in natural frequencies. To do this, a Gaussian process approach was used, avoiding the necessity of too many evaluations as in the Monte Carlo method and allowing a better understanding of the results and more robust models. A plate was used to avoid geometry uncertainties due to its easy modelling and results from* $Ansys^{©}$ *academic mechanical 18.1 simulation were used as reference.*

Keywords: *: Modal Analysis, Uncertainty, Gaussian Process, Simulation, Finite Elements*

1. INTRODUCTION

The uncertainty quantification in modal analysis restricts itself mostly with the operational (Chauhan, 2014), not being taken too much in consideration in the simulation case, inducing the use of high safety factors and price increases in the project even when it may not be necessary. One of the uncertainties aspects of modal analysis is the material properties, whose variations have already been studied in other applications such as asphalt pavements (Caro and Castillo, 2014) and high strength steels (Malpally, 2014). Studying those variations allows one to better quantify how much it can trust a simulation result and what behave it should expect.

The approach used in this work to quantify the uncertainty was the use of a Gaussian process as in (Bilionis and Zabaras, 2011). The idea is to train a model with the calculation results and then generate new results with the trained model, which is normally less expensive to calculate. In this way, we can get good results as in the Monte Carlo method without its cost.

For this study, only the material variation was to be quantified, thus a free thin plate was used, avoiding mostly of the uncertainty for other aspects as mesh simplification from complex geometries or boundary interactions.

2. EXPERIMENTAL MODAL ANALYSIS

Experimental modal analysis has te goal to identify the natural parameters of a structure in order to obtain a mathematical modal which is as close as possible from reality (Maia, 1998). These natural parameters are the modal shapes(r), damping rates(ζ) and natural frequencies(ω), and, with them, one can model the impulse response of a structure both in time and in frequency domain with the respectively formulations (Siemens, 2005):

$$h(t) = \sum_{k=1}^{N}(r_k e^{\lambda_k t} + r_k^* e^{\lambda_k^* t}) \tag{1}$$

$$H(j\omega) = \sum_{k=1}^{N}(\frac{r_k}{(j\omega - \lambda_k)} + \frac{r_k^*}{(j\omega - \lambda_k^*)}) \tag{2}$$

The pole value λ_k can be expressed as:

$$\lambda_k = -\zeta_k\omega_k + j\omega_k\sqrt{1-\zeta_k^2} \tag{3}$$

To obtain those parameters from measured data, some mathematical techniques are needed, and, for a better understand of the reader, two of the main models of modal analysis estimation (one in time and other in frequency domain) that were also used in this work are quickly presented in the next section.

2.1 Main mathematical models

2.1.1 Least-Squares-Complex-Exponential(LSCE)

LSCE is maybe one of the most famous method for obtaining parameters in modal analysis where input and outputs are know. It is a method in the time domain and it is a curve fitting method that uses a auto-regressive moving average(ARMA) model (Kerschen and Golinval, 2004). To use it we have to measured the impulse response for multiple outputs in our structure with one or various inputs.
Taking the IFFT(Inverse Fast Fourier Transform) from the frequency response data and letting $V_k = e^{\lambda_k(j\Delta t)}$, we can write for each r-input,k-output IRF_{rk} the following simplification:

$$h(j\Delta t)_{rs} = 2Re[\sum_{k=1}^{n} A_{rs(k)}V_k^j] \tag{4}$$

Treating the measured IRF as a ARMA model and assuming the force function as being 0 for all times greater than on (Considering a free decay IRF) one ends with the following equation:

$$2Re[\sum_{k=1}^{n} A_{rs(k)} \sum_{j=0}^{o_q} \beta_j V_k^j] = 0 \tag{5}$$

For it to be true, we need that $\sum_{j=0}^{o_q} \beta_j V_k^j$ equals zero, so one can obtain the β coefficients with the IRF and then find the values of V_k and therefore the natural frequencies, damping rates and mode shapes as the eigenvectors of the complementary matrix of the polynomial:

$$\beta_0 + \beta_1 V_k + ... + \beta_{oq} V_k^{oq} = 0 \tag{6}$$

2.1.2 PolyMax

The Polymax is a frequency domain curve fitting method and is based in a model of a right matrix-fraction (Guillaume *et al.*, 2003) as :

$$H(\omega)_o = N(\omega)_o D^{-1}(\omega) \tag{7}$$

for output $o = 1, ..., N_0$ with:

$$D(\omega) = \sum_{j=0}^{n} \Omega_j(\omega) A_j \tag{8}$$

and the Numerator:

$$N(\omega) = \sum_{j=0}^{n} \Omega_j(\omega) B_j \tag{9}$$

The goal of the method is to minimize the $A's$ and $B's$ coefficients,which can be written as the column vector $\theta = [\beta_1, ..., \beta_{No*Nin}, \alpha]^t$ with:

$$\beta_k = B_{k0}, ..., B_{kn}^t, \alpha_k = A_0, ..., A_n^t \tag{10}$$

The function to minimize becomes then:

$$Error(\theta) = \sum_{k=1}^{N_o N_f} \sum_{f=1}^{N_f} |W_k(w_f)(\frac{N_k(w_f), \beta_k}{d(w_f, \alpha)}) - H_k(w_f)|^2 \tag{11}$$

Where W_k is a weighting function proportional to the inverse square of the variance of the measurements in a determined frequency. This function is minimized equating the gradient to zero and the natural parameters are calculated in a similar way as the LSCE, using the complementary matrix in the α Coefficients. Besides those two presented methods, there are many others to the calculation of modal parameters, examples being the Inverse Method, Dobson, single least-squares and Eingensystem Realization, which were all used in the presented work but their detailed explanation would not increment its general understanding. For the more advance reader some good references are (Pinheiro, 2015), (Juang and Pappa, 1986).

2.2 Thin Plate with Finite Element Method

The Kirchhoff plate theory (Ramu and Mohanty, 2012) was used to model a four node rectangular element to solve in the computational method. The element has three degrees of freedom(DOF) in each node, being 12 DOF in total, the values to DOF are:

$$w; \theta_x = \frac{1}{b}\frac{\partial w}{\partial \eta}; \theta_y = \frac{1}{a}\frac{\partial w}{\partial \varepsilon} \tag{12}$$

A representation of the element can be seen in the figure 1 below:

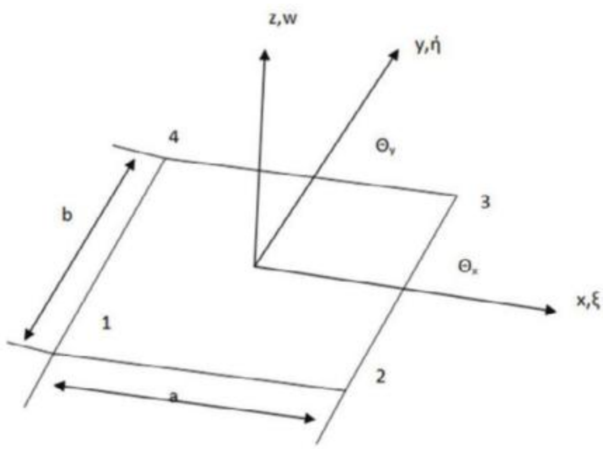

Figure 1. Plate Element used. Source:(Ramu and Mohanty, 2012)

The displacement w can be written as:

$$w(\eta,\varepsilon) = a_1 + a_2\eta + a_3\varepsilon + a_4\eta^2 + a_5\eta\varepsilon + a_6\varepsilon^2 + a_7\eta 3\varepsilon + a_8\eta^2\varepsilon + a_9\eta\varepsilon^2 + a_{10}\varepsilon^3 + a_{11}\eta^3\varepsilon + a_{12}\eta\varepsilon^3 \tag{13}$$

$$w(\eta,\varepsilon) = \{p(\eta,\varepsilon)\}^T\{\alpha\} \tag{14}$$

To write this equation in function of the degrees of freedom, it is possible to evaluate $w(\eta,\varepsilon)$ and its partials in $\eta = \pm 1$ and $\varepsilon = \pm 1$ in order to get a element matrix $[A_e]$ so to write:

$$w_e = [A_e]\{\alpha\} \tag{15}$$

Using the principle of virtual work one can then get:

$$T_e = \frac{1}{2}\{\dot{w}_e\}^T[M_e]\{\dot{w}_e\} \tag{16}$$

$$U_e = \frac{1}{2}\{w_e\}^T[K_e]\{w_e\} \tag{17}$$

The mass and stiffness matrix for each element become:

$$[M_e] = [A_e]^{-T}[\int_{-1}^{1}\int_{-1}^{1} \rho h\{p(\eta,\varepsilon)\}^T\{p(\eta,\varepsilon)\}abd\eta d\varepsilon][A_e]^T \tag{18}$$

$$[K_e] = [A_e]^{-T}[\int_{-1}^{1}\int_{-1}^{1} I_z\{p^{''}(\eta,\varepsilon)\}^T[D]\{p^{''}(\eta,\varepsilon)\}abd\eta d\varepsilon][A_e]^T \tag{19}$$

$$p^{''}(\eta, \varepsilon) = [\frac{1}{a^2}\frac{\partial^2}{\partial \eta^2}p(\eta, \varepsilon)\}^T, \frac{1}{b^2}\frac{\partial^2}{\partial \varepsilon^2}p(\eta, \varepsilon)\}^T, \frac{1}{ab}\frac{\partial^2}{\partial \eta \partial \varepsilon}p(\eta, \varepsilon)\}^T] \quad (20)$$

h is the thickness, ρ is the density, I_z is the moment of inertia and $[D]$ is the matrix that relates deformation and strain and its given by:

$$[D] = \frac{E}{1-\upsilon^2}[1, \upsilon, 0; \upsilon, 1, 0; 0, 0, \frac{1-\upsilon}{2}] \quad (21)$$

When using the finite element method to dynamics the main movement equation can be described using a mass, damping and stiffness matrix, as well as a force vector (Ramu and Mohanty, 2012). This equation can be seen below:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}(t)\} + [K]\{x(t)\} = f(t) \quad (22)$$

To get the natural frequencies the homogeneous problem is solvend considering a harmonic response. In this way one ends with the following equation:

$$([K] - \omega^2[M])x = 0 \quad (23)$$

which is only possible when:

$$det([K] - \omega^2[M]) = 0 \quad (24)$$

Solving the generalized eigenvalue problem one obtain the eigenvalues $\omega's$ and the mode shapes as the eigenvectors.

2.3 SMA

To the calculation of the modal parameters, a program developed by the author was used, it is called Sma (Smart Modal Analysis), it was developed in the Python language with the Anaconda distribution (Analytics, 2016) and it aims to calculate modal parameters with a variety of methods. The main interface of the program can be seen in Figure 2:

Figure 2. Sma main user interface.

In it one can load all the measured data and choose which method to use in the calculation, both in time and frequency domain. The user is also allowed to give the positions of the measurements and automatically identify from which input a specific measurement comes in respect of its name. Having many methods bounded in only one program allows a evaluation also of the uncertainty that comes with the data process itself. The methods used in this work to evaluate the natural frequencies were: Polymax, LSCE, Improved Least-Squares, Inverse Method,Dobson and Eigensystem Realization.

3. GAUSSIAN PROCESS

A Gaussian process is a non-parametric kernel based regression that aims to predict a response variable from given a determined input (Bilionis and Zabaras, 2011).
With a vector input X and the relation $X = y = g(X)$ A Gaussian process can be modelled as follows:

$$G(X) = f(X)^T \beta + Z(X) \quad (25)$$

where $f(X)^T \beta$ is a linear regression model and $Z(X)$ is a zero-mean Gaussian process with the covariance function:

$$C(X, X') = \sigma^2 R(|X - X'|) \quad (26)$$

σ^2 is the variance and R is the correlation function that depends of the absolute relative distance of the samples.

The best linear unbiased prediction (Buitinck *et al.*, 2013) is chosen giving its linearity,unbiased and best property, which imply:

$$\hat{G}(X) = a(X)^T y \quad (27)$$

$$E[G(X) - \hat{G}(X)] = 0 \quad (28)$$

$$\hat{G}(X)^* = argminE[(G(X) - \hat{G}(X))^2] \quad (29)$$

One then can find the optimal weight vector as:

$$a(X)^* = argmin(E[(G(X) - a(X)^T y)^2]) \quad (30)$$

with the condition that $E[G(X) - a(X)^T y] = 0$.

4. EXPERIMENTAL PROCEDURE AND METHODOLOGY

The evaluation was divided in two parts, experimental and simulation. In the first one a experimental modal analysis of a thin steel plate with dimensions 36 cm x 41.6 cm x 0.27 cm suspended with two nylon strings to simulate a free plate vibration, which can be seen in the figure 3.

Figure 3. Thin steel plate suspended and the accelerometer used.

The equipments and software used for the experimental part were:
- Accelerometer PCB 353B16;
- Impact hammer PCB 086C04;
- Pulse B&K 7536;
- Pulse LabShop 12.5

- Thin Steel plate

The frequency response in the point right where was the accelerometer showing some of the resonance peaks can be seen in the figure 4 below:

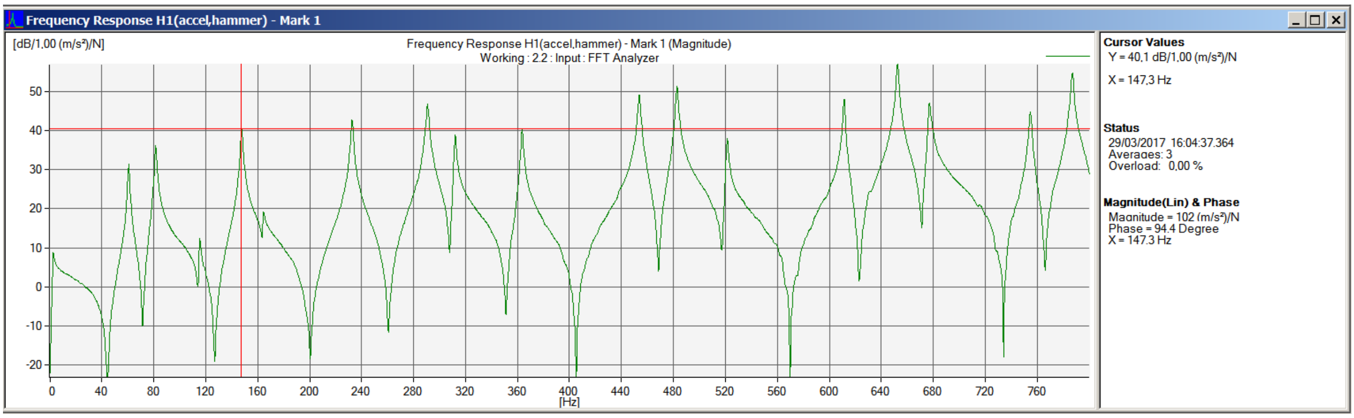

Figure 4. Frequency response of a thin plate in the experimental modal analysis.

The data was then processed in the Sma software to obtain the natural frequencies and their uncertainty within the methods. In this work only the natural frequency uncertainty was evaluate, because the use of proportional damping could mask the results since its coefficients are optimized with the measurement data.

To do the simulation, the finite element method approach was used to simulate the thin plate and get the results. The element used was described in the finite elements section and it was choose to simulate both out of the plane motion and translational movement, the used mesh can be seen in the figure 5 below and its number of elements was refined until there was no considerable chance in the results.

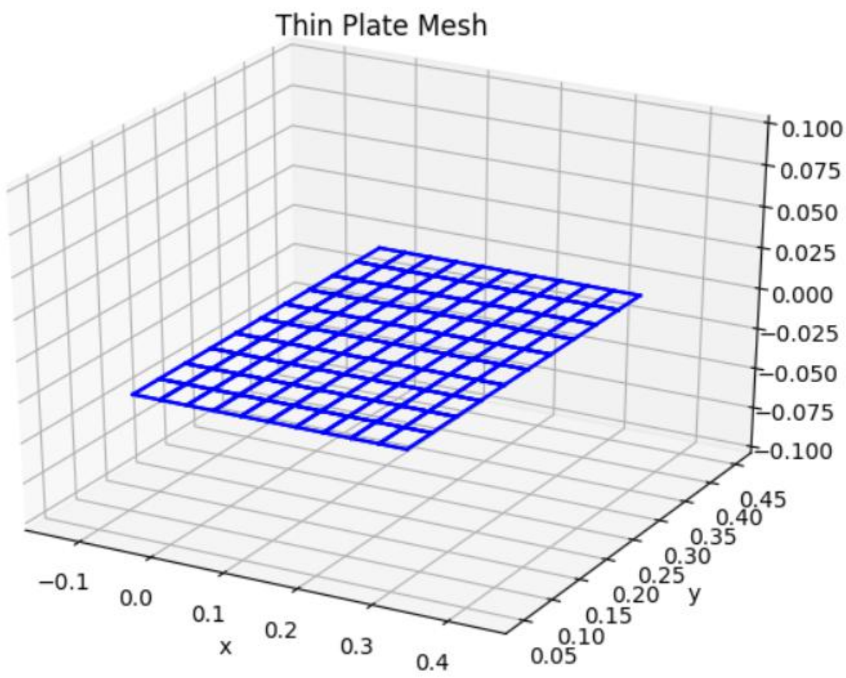

Figure 5. Used mesh.

The frequencies were calculated for a steel with $E = 200GPA$, $v = 0.3$ and $\rho = 7850kg/m3$. Those parameters were calculated with a 5% noise variance in a random fashion, so:

$$X(E, v, \rho) = X(E, v, \rho) + 0.05X(E, v, \rho) \times N(-1, 1) \tag{31}$$

Where $N(0, 1)$ is a uniform distribution between -1 and 1. We can then make a Monte Carlo evaluation and use the partial results to train a Gaussian process with was implemented in the scikit-learn library (Buitinck *et al.*, 2013). The finite element calculation was coded in Python.

5. RESULTS AND DISCUSSIONS

In the figure 6 below we see the first and fourth natural frequencies from a sample of 5000 evaluations with the random parameter variance together with the experimental one and Ansys© (ANSYS, 2017) as reference:

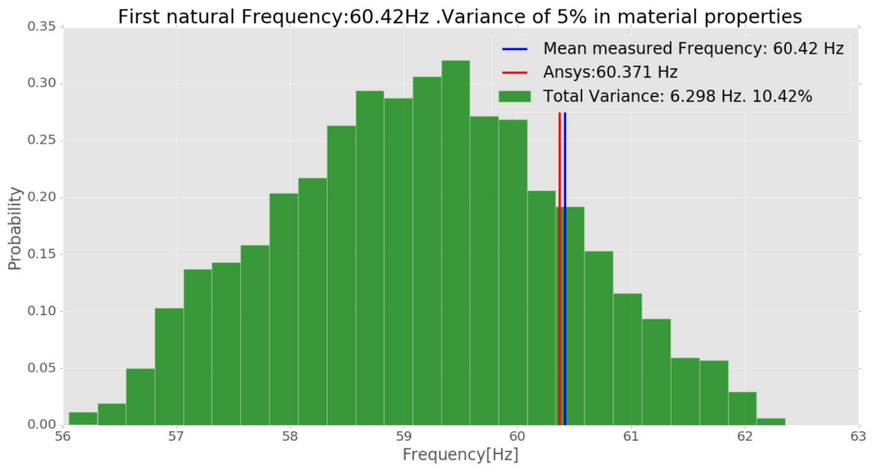

Figure 6. Variation in first natural frequency due to material Uncertainty

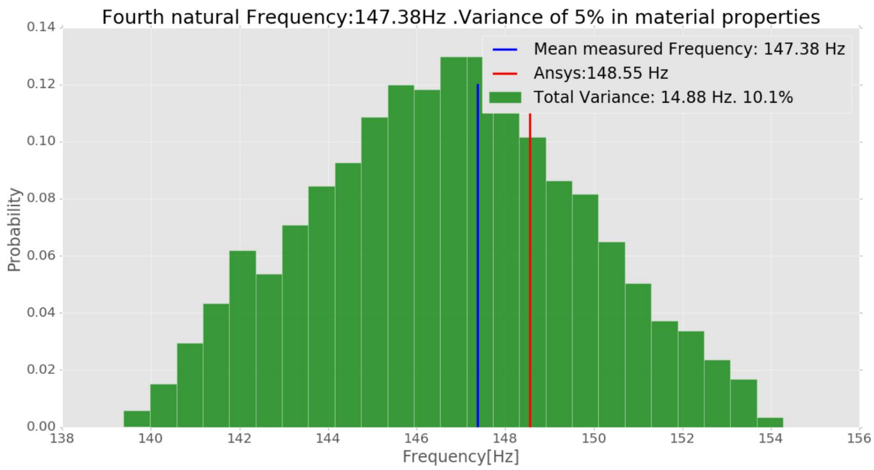

Figure 7. Variation in fourth natural frequency due to material Uncertainty

We see in both cases a total variation from roughly 10%, which is quite a few. Now we compared the uncertainty from the material and from the different methods, to verify if we would expect a significant increase uncertainty due to lack of exact material knowledge. We can see for the first seven natural frequency the standard deviation from the methods and from the material uncertainty in Figure 8.

It becomes clear that the material uncertainty is extremely more significant than the measurement one.

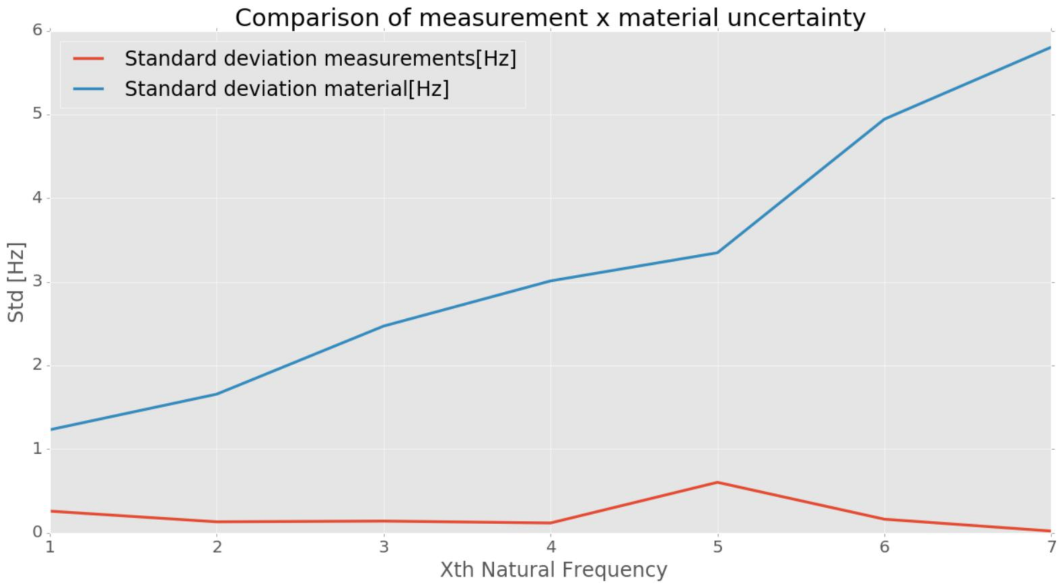

Figure 8. Measurements x Material uncertainty for first seven natural Frequencies

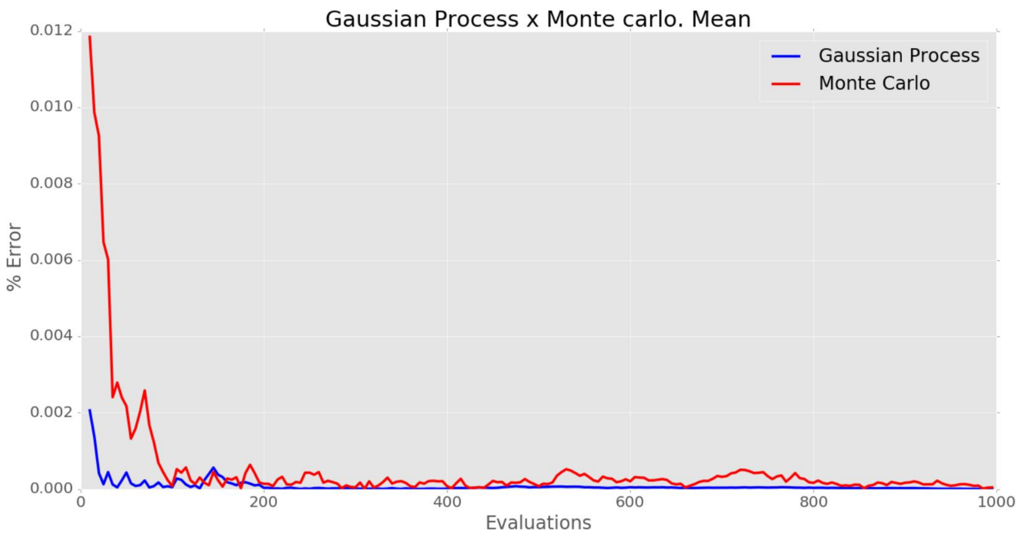

Figure 9. Convergence of mean Gaussian Process x Monte Carlo.

A convergence to the deviation both for Gaussian process and Monte Carlo was evaluated. The result of the 1st natural frequency can be seen in Figure 9 and 10 for the first 1000 evaluations.

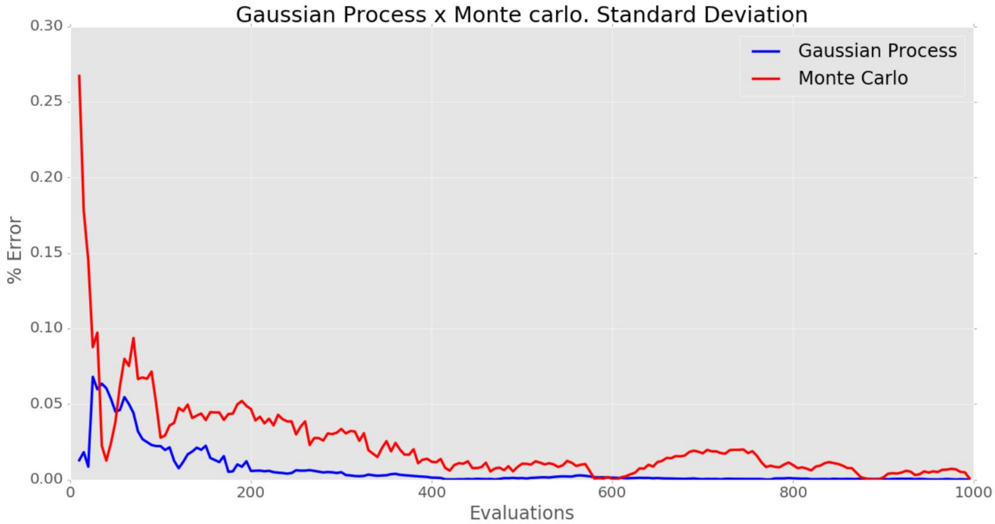

Figure 10. Convergence of standard deviation Gaussian Process x Monte Carlo.

We can see that the Gaussian process was able to converge extremely faster than the Monte Carlo in the standard deviation and mean, which is a tremendous advantage when we have a complex to evaluate model. The pattern seen in the first node is similar to all other frequencies. A table with the values for the 4 first natural frequencies can be seen in the Table 1:

Table 1. First Natural Frequencies

Natural Frequency[Hz]	Mean Measured	Ansys	Mean/std
First	60.42	60.37	59.25 ± 1.24
Second	81.29	81.72	80.67 ± 1.6
Third	114.92	117.64	117.1 ± 2.4
Fourth	147.37	148.55	146.89 ± 3.03

6. CONCLUSION

We were able to see that our model got all the measured frequencies in the uncertainty range of one standard deviation and that the Gaussian process was able to converge way faster and with less interactions than the pure Monte Carlo method, proving its computational efficiency. That efficiency can be extremely helpful when evaluation complex systems where one does not have the time to run thousands of interactions but still wants to get a accurate quantification of the problem uncertainty.

The used method is very robust and could be used to any problem whatsoever that has some unknown or uncertain variable. Future works are going to be made to evaluate the uncertainty of complex system with the proposed Gaussian process approach.

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8. RESPONSIBILITY NOTICE

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