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ADAPTIVE ENRICHMENT OF THE GENERALIZED FINITE ELEMENT METHOD BASED ON THE FRIBERG ERROR INDICATOR

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Abstract.
The Generalized Finite Element Method (GFEM) has presented excellent results when applied to free vibration analysis of structures mainly when obtaining higher frequencies. However the classical GFEM enriching approaches apply the enrichment over all elements in the mesh increasing a lot the computational cost. This work proposes an adaptive enrichment of the GFEM based on the Friberg error estimator. This technique allows choosing the elements to enrich that better increase the precision of a specific frequency. The application of this approach in free vibration analysis of bars is presented. A brief explanation about the p-hierarchical FEM and the GFEM, and about how the estimator works is given. Finally the results obtained by the adaptive enrichment of GFEM are compared with those obtained by analytical solutions and classical refinements of FEM and GFEM.
Keywords: *Generalized Finite Element Method, error estimator, free vibration analysis.*

1. INTRODUCTION

The computational methods are very important for the resolution of several engineering problems, because allow to obtain the approximate solutions for the analysis of practical problems that were previously impossible to solve due to the high degree of complexity they have. The search for reliable methods for the resolution of contour value problems is a recurrent theme in the literature and with the computational advances there has been a significant improvement in terms of accuracy and convergence (Torii, 2012).

Determining the natural frequencies of a structure can involve a very high computational cost, even using established methods such as the Finite Element Method (FEM) and the Generalized Finite Element Method (GFEM), since these methods solve eigenvalue problems which size depends on the discretization of the problem domain (*h* refinement) and the number of shape functions (*p* refinement).

The Generalized Finite Element Method (GFEM) has presented excellent results when applied to free vibration analysis of structures mainly when obtaining higher frequencies. Some recent works GFEM are Arndt (2009), Babuska and Banerjee (2012), Duarte (2012), Torii and Machado (2012), Shang (2014), Weinhardt (2016) and Petroli (2016). However the classical GFEM enriching approaches apply the enrichment over all elements in the mesh increasing a lot the computational cost.

It is extremely important to know how far the approach found by the FEM and GFEM lies in the real solution of the problem, since there are several types of errors that can compromise the approximate solution, such as discretization errors, integration errors, rounding errors and propagation, so a solution to work around this problem and give more credibility to the analysis are the error estimators. According to Gratsch and Bathe (2005), there are two types of error estimators, the *a priori* error estimators, which are applied before the problem is solved, and the *a posteriori* error estimators, which are applied after approximation is calculated.

In the work of Lins (2011) an error estimator a posteriori for the GFEM, called EPMEFG, is proposed. This estimator uses least squares in relation to the points of super-convergence, based on the recovery of nodal values of the solution gradients. In Leite (2016) a study where is presented an upper limit of the discretization error in the Method of Spectral Elements is performed on the wave equation with variable coefficients. Another work (Silva, 2015)about performs the analysis of the main *a posteriori* error estimators applied to the FEM using *h*-adaptive refinement, which is a mesh refinement using pre-established criteria. In this work the error estimators are based on the recovery of the upper derivatives and the energy norm.

In this paper an adaptive enrichment approach based on the Friberg *a posteriori* error indicator (η_i), which was presented by Friberg (1986) and also discussed in Friberg and Moller (1987), is proposed in order to solve eigenvalue problem from free vibration analysis of structures precisely with lower computational cost. In the adaptive process the GFEM hierarchical *p* refinement is adopted in order to use mass and stiffness matrices of a previous level of enrichment in the construction of a new level (Zienkiewicz and Taylor, 1977). With the use of the estimator it is possible to identify which element carries the most error during the resolution process of the eigenvalue problem, focusing attention on the elements that most impair the accuracy of the solution, in order to reduce the error and the dimension of the matrices involved (Duarte, 2003). As the Friberg indicator serves to make a dimensionless projection of an eigenvalue, when the system grows more degrees of freedom, without the new system being solved, then the indicator also presents characteristics of the *a priori* error estimators.

2. DYNAMIC BAR ANALYSIS WITH THE FEM AND GFEM WITH THE FRIBERG INDICATOR

The problem of a straight bar with longitudinal vibration is described by the partial differential equation expressed by

$$\rho A \frac{\partial^2 \bar{u}}{\partial t^2} - \frac{\partial}{\partial x}\left(EA\frac{\partial \bar{u}}{\partial x}\right) = p(x,t) \tag{1}$$

subject to the boundary conditions, where ρ is the specific mass, E is the modulus of elasticity, A is the cross-sectional area and $p(x,t)$ is the axial force applied. This problem comes down to finding an approximate solution $\bar{u}(x,t)$ so that equality is maintained. If E, A and ρ are constants and force $p(x,t) = 0$, which characterizes a free vibration problem, the Eq. (1) becomes in:

$$\rho A \frac{\partial^2 \bar{u}}{\partial t^2} - EA\frac{\partial^2 \bar{u}}{\partial x^2} = 0 \tag{2}$$

In this paper, one the cases that we discuss is of a fixed-free uniform bar, which is illustrated by the Fig. 1.

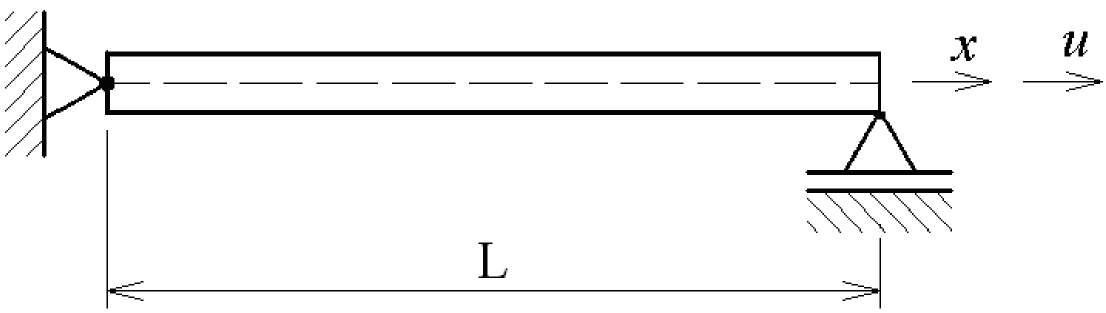

Figure 1. Straight bar fixed-free with axial deformation.
(Arndt, 2009)

The mass matrix $[M^e]$ and the elementary stiffness matrix $[K^e]$ of the FEM are constructed by applying the Galerkin method in the weak form of Eq. (2) and are given by

$$[K^e] = EA\int \frac{d\theta_i}{dx}\frac{d\theta_j}{dx}\,dx \tag{3}$$

$$[M^e] = \rho A\int \theta_i\theta_j\,dx \tag{4}$$

where θ_i and θ_j are shape functions, which in FEM are generally polynomial. The linear FEM shape functions defined for $x = [0,1]$ are shown in Fig. 2.

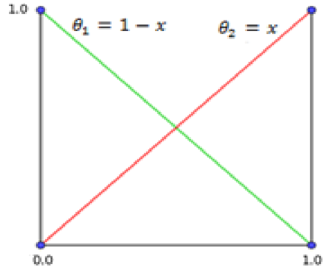

Figure 2. Linear shape functions.

When the global mass and global stiffness matrices of dimensions n x n are constructed it is possible to rewrite the equilibrium equation of the problem so that it becomes a generalized eigenvalue problem given by

$$([M]_{nn} - \lambda_i^n . [K]_{nn})\{\Phi_i^n\} = 0 \tag{5}$$

which results in approximations for the eigenvalues λ_i^n and for the associated modes Φ_i^n. The natural frequencies of the structure are calculated by $\omega_i = (\lambda_i)^{0.5}$.

According to Szabo and Babuska (1991) the analytical solution of practical problems is rarely known. Thus there are two techniques to increase the precision of the approach: an *h* refinement or a *p* refinement. The *h* refinement of the finite element implies in a new calculation of the mass and stiffness matrices. The *p* refinement is performed by the addition of new shape functions. This process can be done hierarchically, where the mass and stiffness matrices used to approximate ω_i and ϕ_i are submatrices of the new mass and stiffness matrices, so it is not necessary to recalculate the whole system, but only part of it. This hierarchical approach generates a reduction in computational cost.

The GFEM is a method that has the property of inserting known characteristics of the solution of the problem into the approximation space (Babuska and Banerjee, 2012). In this work we use the GFEM with the trigonometric enrichment functions proposed by Arndt (2009). These enrichment functions (Fig. 3) are expressed in the interval $x = [0,1]$, by:

$$\phi_3 = \sin\ (\beta_j . L_e . x) \tag{6}$$

$$\phi_4 = \left(\cos\ (\beta_j . L_e . x) - 1\right) \tag{7}$$

$$\phi_5 = \sin\ (\beta_j . L_e . (x - 1)) \tag{8}$$

$$\phi_6 = \left(\cos\ (\beta_j . L_e . (x - 1)) - 1\right) \tag{9}$$

where L_e is the element length and $\beta_j = 1\pi, 2\pi, 3\pi, 4\pi, ...$ is a parameter used to construct different shape functions. Before performing the enrichment these functions are multiplied by the linear form functions of Fig. 2 so that they are a unit partition and present compact support in the element domain.

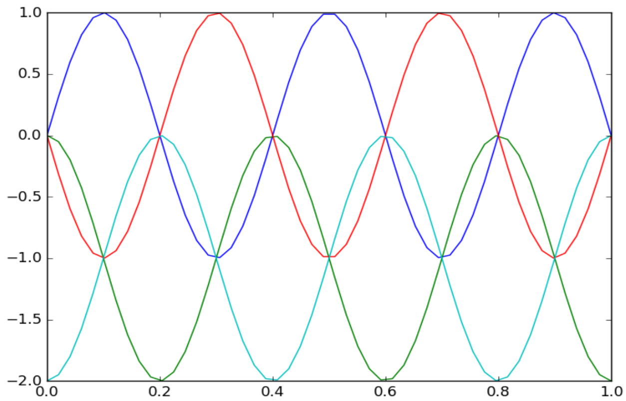

Figure 3. GFEM enrichment functions with $\boldsymbol{\beta_j} = 5$.

Since the fixed-free straight bar vibration problem has the known analytical solution, the frequency j can be calculated by

$$\omega_j = \frac{(2j-1)\pi}{2L}\sqrt{\frac{E}{\rho}} \tag{10}$$

When the approximation of a frequency ω_i is calculated, the relative error is calculated with respect to the analytical solution ω_j, according to the expression

$$\text{Relative Error} = \frac{\omega_i - \omega_j}{\omega_j} \tag{11}$$

When the analytical solution is not known, it becomes impossible to calculate the relative error with Eq. (11), but the work of Friberg (1986) proposes an error indicator based on the relative error for eigenvalues. This indicator is able to approximate the relative variation in a given eigenvalue λ_i^n by adding new m form functions in the system of dimension n, written in the form of Eq. (5) This new hierarchical system of dimension $(n + m)$ is written by:

$$\left(\begin{bmatrix} [M]_{nn} & [M]_{nm} \\ [M]_{mn} & [M]_{mm} \end{bmatrix} - \lambda_i^{n+m}.\begin{bmatrix} [K]_{nn} & [K]_{nm} \\ [K]_{mn} & [K]_{mm} \end{bmatrix}\right)\begin{Bmatrix} \Phi_i^n \\ \Phi_i^m \end{Bmatrix} = 0 \tag{13}$$

Since the problem of n-dimensional eigenvalues is solved, so λ_i^n and Φ_i^n are known, starting from the Rayleigh Quotient and an expansion of the Taylor Series in 2D, it is possible to approximate the relative error of the eigenvalue by (Duarte, 2003):

$$\eta_i = \frac{\lambda_i^{n+m} - \lambda_i^n}{\lambda_i^n} \approx \frac{\{\Phi_i^t\}.\left[[K]_{mn} - \lambda_i^n.[M]_{mn}\right]^t.\left[\left[[K]_{mm} - \lambda_i^n.[M]_{mm}\right]^{-1}\right]^t.\left[[K]_{mn} - \lambda_i^n.[M]_{mn}\right]\{\Phi_i\}}{\{\Phi_i^t\}[K]_{nn}\{\Phi_i\}} \tag{14}$$

However, in this work, Eq. (14) is used to predict variations in eigenvalues caused by the application of GFEM in a finite element mesh that had the initial solution obtained by Linear FEM. The main advantage of the use of the Friberg indicator is that the new system does not have to be solved in order to have an idea of the eigenvalue variation between different GFEM approaches. However it is necessary to be careful because the values of the indicator are dimensionless, even negative, and cannot be used to make extrapolations for the eigenvalues. . The value of the relative error indicator for the eigenvalue λ_i^n can be extended to the natural frequency ω_i^n by

$$\eta_i = \frac{\lambda_i^{n+m} - \lambda_i^n}{\lambda_i^n} = \frac{(\omega_i^{n+m})^2 - (\omega_i^n)^2}{(\omega_i^n)^2} \tag{15}$$

3. RESULTS AND DISCUSSIONS

The computational procedure consists in applying the proposed adaptive GFEM to vibration analysis of bars. The frequencies and vibration modes obtained are compared to those obtained by analytical solution and classical refinements of FEM and GFEM. The computational effort necessary to perform each of one analysis is also compared, based on the number of degrees of freedom of the problem.

Three cases were studied: the first is a uniform fixed-free bar discretized by a distorted mesh, which was proposed by Duarte (2003) in order to check if the Friberg error indicator can estimate the contribution of each element of the mesh; the second case consists of the same bar discretized by a uniform mesh whose objective is to improve the accuracy of the approximation; and a third simulation of a bar formed by two different materials in order to know if there are regions of interest.

The numerical simulation was performed in Python version 2.7, because it is a high level computational language that facilitates the implementation of the problem and produces an appreciable computational efficiency. In this work the unit of natural frequencies is rad/s.

3.1 Uniform bar with distorted mesh

The adaptive refinement applied to the GFEM free vibration analysis of a simple clamped free bar discretized by a non-uniform mesh is now presented in order to illustrate its efficiency. The bar has the following characteristics: length

1 m, modulus of elasticity 1 N/m^2 , cross-sectional area 1 m^2 and mass density 1 kg/m^3. The bar is divided into five elements with lengths 0.025 m, 0.075 m, 0.2 m, 0.3 m and 0.4 m. When applying the Linear FEM with five elements, one has a system with five degrees of freedom and in solving it, one obtains approximations for the first five bar frequencies. As the analytical solution is known the relative error is calculated according to Eq. (11) and the results are given in Tab. 1.

Table 1.Linear FEM approaches for the first five frequencies.

Frequency	Linear FEM	Analytical	Relative Error (%)
ω_1	1.59333752	1.57079632	1.43%
ω_2	5.11174445	4.71238898	8.47%
ω_3	8.68294863	7.85398163	10.55%
ω_4	14.64112604	10.99557429	33.15%
ω_5	43.54426903	14.13716694	208.01%

The approximations presented in Tab. 1 show that the greatest error occurs in obtaining the frequency ω_5. Starting from the fact that the GFEM is a good tool to approximate the higher frequencies, the GFEM with $\beta_j = 1\pi$, is applied in only one of the mesh elements and four new field degrees of freedom are inserted, resulting in a system with nine degrees of freedom .

Then the Friberg indicator is calculated having a target frequency ω_5. The value found for the indicator indicates a dimensionless projection of the variation generated in the fifth frequency when solving the system with nine degrees of freedom, without the need to solve it. However in this work the problem is solved again so that there is numerical proof that the Friberg indicator values make sense.

Then, the enrichment functions applied in the previous element are removed and enrichment is performed on another element of the mesh, followed again by the calculation of the Friberg indicator and the solution of the new system with nine degrees of freedom.

This procedure is repeated until all the elements of the mesh receive the enrichment in order to compare the values of the indicator with the objective of finding the element or regions of the mesh where the enrichment causes the greatest convergence of the solution. The highest value of the indicator corresponds to the greatest variation in the target frequency. The values of the indicator, the approximations for ω_5 and the relative error calculated with the value of the analytical solution are given in Tab. 2.

Table 2.Aproximations of the fifth frequency with the GFEM in only one element – Distorted mesh.

Element	η_i	ω_5	Relative Error (%)
L_1	-0.0812	43.06983005	204.65%
L_2	-3.2758	36.28848518	156.68%
L_3	-0.0986	23.64230512	67.23%
L_4	-0.0027	17.50023238	23.78%
L_5	-0.0011	15.35714856	8.62%

It is observed that the approximate frequencies vary from element to element enrichment, where the best approximation corresponds to a relative error of 8.62% and is obtained when the fifth element receives the four GFEM enrichment functions. The values of the indicator successfully represent the variation of the frequency ω_5 in four elements of the mesh, however in the first element the value of the indicator is greater than it should be, because when this element receives enrichment functions, the new approximation contains an error of 204.65%. This occurs because the element is too small and it is on the contour of the bar producing a mass matrix close to a nondefined positive matrix and affecting the calculation of the indicator. Although the L_1 element is disregarded for enrichment, this fact does not impair the calculation of the indicator and the analysis in the other four elements. The relationship between the indicator values and the relative error of the target frequency is shown in Fig. 4.

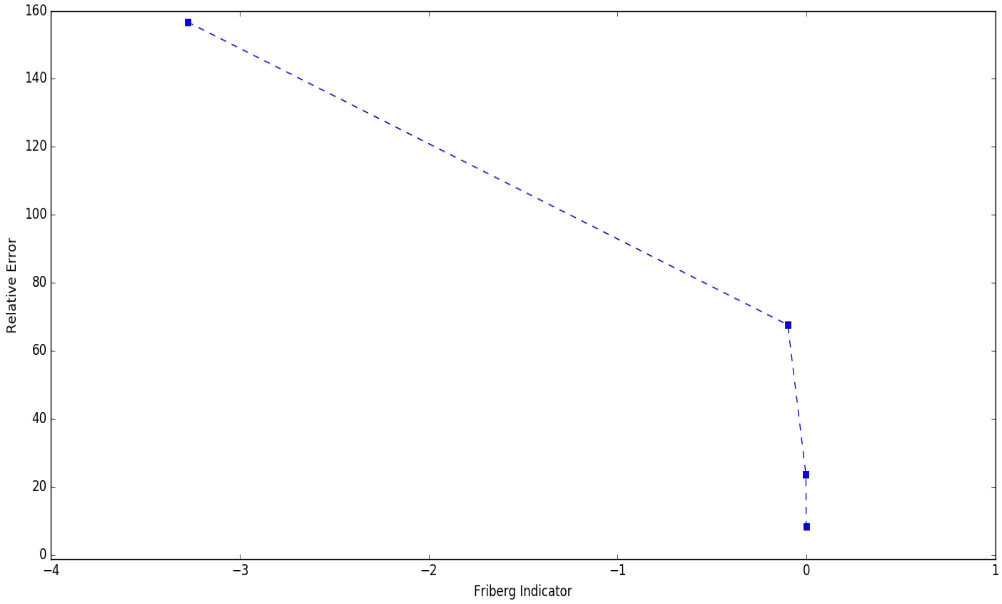

Figure 4. Relation between the indicator and the relative error.

According to Fig. 4, it is observed that the larger the value of the indicator, the smaller the relative error obtained by the approach.

In Table 3 the fifth frequency obtained by the GFEM refining just the element L_5 is compared with those obtained by Linear FEM and *p*-versions of FEM.

Table 3. Method comparisons.

Method	ω_5	ndof	Relative Error (%)
Linear FEM	43.54426903	5	208.01%
FEM-p	16.71510680	6	18.23%
FEM-p	16.27062431	10	15.09%
GFEM	15.35714856	9	8.62%

It is observed that the approximation obtained with the GFEM presents an error of 8.62% for a system with nine degrees of freedom, five referring to the FEM-linear and four referring to the four trigonometric functions that are all placed in the fifth element. The approximation with the ten degrees of freedom FEM was taken from Duarte (2003) for comparison, and presents 15.09% error for a system with ten degrees of freedom, five of which refer to FEM-linear and five referents to the FEM-p, where each element of the mesh receives two functions. The approximation of the FEM-p with six degrees of freedom presents an error of 18.23%, where five degrees of freedom are FEM-linear and a degree of freedom is added when only the fifth element receives a new shape function of the FEM-p.

The success of the Friberg indicator is evident, since it allows to choose the element in the mesh that shall be refined in order to better improve the convergence of a specific frequency. Besides that, the application of the Friberg indicator in GFEM allows to obtain a fifth frequency more precise than that obtained by FEM-p even with more degrees of freedom.

The error of the approximations obtained with the FEM-p with six degrees of freedom and with ten degrees of freedom has a decay of 3.14%. This is little compared to the decay of the relative error between the approximations obtained with the Liner-FEM with five degrees of freedom and the FEM-p with six degrees of freedom, which is 189.78%, which represents a considerable gain in accuracy, leading to the fact that only one degree of freedom is added to the system.

Finally, the enrichment of all elements of the mesh is carried out following the order of the Friberg indicator, but without removing the functions when another element of the mesh is enriched, so the convergence can be related to the increase in the number of degrees of freedom, according to Fig. 5.

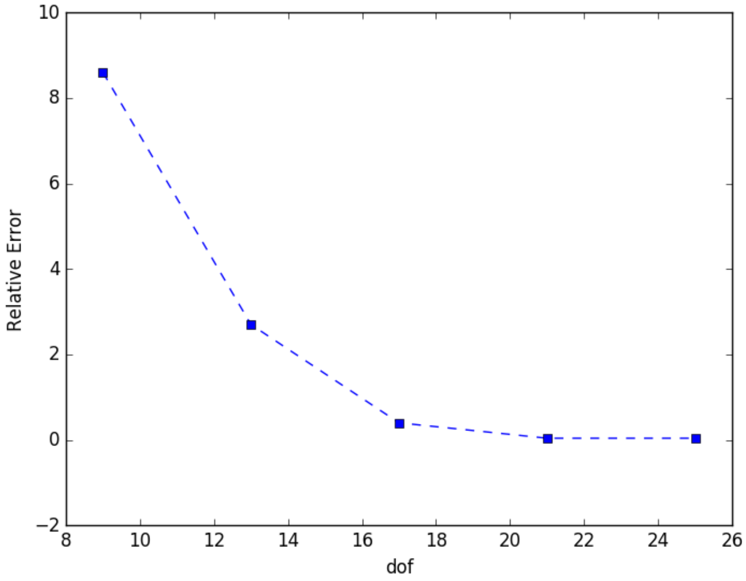

Figure 5. Convergence by adding degrees of freedom.

It is observed that the values of the indicator pointed correctly where the convergence would be greater, since the convergence decays monotonically when more degrees of freedom are inserted following the order of the indicator. It can be seen too that the solution of a system with seventeen degrees of freedom is very close to the solution of a system with twenty five degrees of freedom, which can reduce the overall system by 32%, if the smaller system is chosen. Without the use of the indicator, the GFEM would be applied in a conventional way and it would not be possible to perceive the decay of the convergence that is verified in Fig.5.

3.2 Uniform bar and uniform mesh

The bar used is the same as in simulation 3.1, but the discretization loop is composed of five elements, all with length $L_e = 0.2\ m$. As soon as the linear FEM is applied, the approximations for the first five frequencies are obtained (Tab.4).

Table 4.Linear FEM approaches for the first five frequencies.

Frequency	Linear-FEM	Analytical	Relative Error (%)
ω_1	1.57726369	1.57079632	0.41%
ω_2	4.88813594	4.71238898	3.72%
ω_3	8.66025404	7.85398163	10.26%
ω_4	12.98647071	10.9955742	18.10%
ω_5	16.70338571	14.1371669	18.15%

One can observe that the five approximations are better than those obtained with a distorted mesh, whose results are presented in Tab. 1. The fifth frequency approximation presented the greatest variation, from a relative error of 208.01% to 18.15%. Despite the success obtained with the use of a uniform mesh, there are numerous cases where there is no possible to use a mesh of elements with same size due to the variation of geometry or mechanical properties.

The fifth frequency, which presents the greatest error, was chosen again as the target frequency in the application of the GFEM associated with the Friberg indicator, as later made for the distorted mesh. The results are showed in Tab. 5.

Table 5.Aproximations of the fifth frequency with the GFEM in only one element – Uniform mesh.

Element	η_i	ω_5	Relative Error (%)
L_1	-0.1887	16.32819986	15.49%
L_2	-0.1359	16.28461808	15.18%

L_3	-0.0505	16.19479324	14.55%
L_4	0.0348	16.12855643	14.08%
L_5	0.0876	16.11457361	13.98%

The values of the Friberg indicator indicate exactly the order of contribution of each element, since the fifth element presents the highest value of the indicator and the best approximation. Besides that, the first element presents the lowest value of the indicator and the worst approximation. However the five values of the indicator show a maximum variation of 0.2763 in absolute values that means that the contribution of each element is similar, how can be proven by the similar relative errors.

Conventionally the GFEM is applied with $\beta_j = 1\pi$ in all elements of the mesh. In order to verify if it is possible to choose β_j based on the Friberg indicator, a selective GFEM analysis is performed applying $\beta_j = 1\pi$ to 5π following the decreasing order of the indicator. The results obtained by Conventional GFEM and Selective GFEM are presented in Tab. 6.

Table 6. Conventional GFEM and selective GFEM

	Conventional GFEM	Selective GFEM
Element	β_j	β_j
L_1	$1\,\pi$	$5\,\pi$
L_2	$1\,\pi$	$4\,\pi$
L_3	$1\,\pi$	$3\,\pi$
L_4	$1\,\pi$	$2\,\pi$
L_5	$1\,\pi$	$1\,\pi$
Frequency (ω_5)	14.13718277	14.13716831
Relative Error (%)	0.000595%	0.000009%

The approximation of the fifth frequency with the Selective GFEM is more accurate than the conventional GFEM, and this is due to the use of the Friberg indicator. This approach is interesting once certain elements contribute more to the convergence of a certain frequency than others.

3.3 Fixed-free bar with two different materials

In the dynamic analysis of structures several times the structures are not made of the same material. In this context the simulation of a bar formed by two different materials, fixed at one end and free at the other end, as shown in Fig.6, is performed.

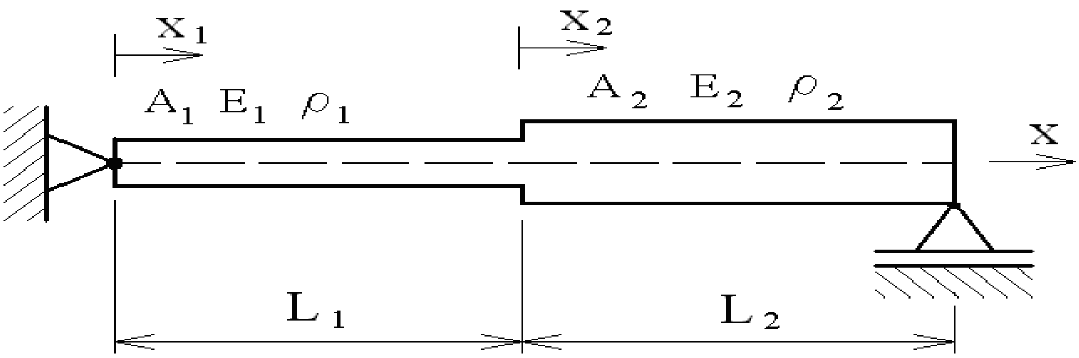

Figure 6. Fixed-free bar with two different materials.
(Arndt, 2009)

The bar has the following characteristics: length $1\,m$, modulus of elasticity $E_1 = 1\ N/m^2$ and $E_2 = 2\ N/m^2$, cross-sectional area $A_1 = 1\ m^2$ and $A_2 = 2\ m^2$, and mass density $\rho_1 = 1\ kg/m^3$ and $\rho_2 = 8\ kg/m^3$. The bar is divided into two elements with lengths $L_1 = 0.5\ m$ and $L_2 = 0.5\ m$. When applying Linear FEM on the bar, one has a system with two degrees of freedom. Tab. 7 shows the first two frequencies obtained analytically (Arndt, 2009) and by Linear FEM.

Table 7. Conventional GFEM and selective GFEM

Frequency	Linear FEM	Analytical	Relative Error (%)
ω_1	0.24013389	0.23794112	0.92%
ω_2	1.73206715	1.57079632	10.26%

Then the GEFM with $\beta_j = 1\pi$ is applied to one element at a time and the Friberg indicator is calculated having a target frequency ω_2, since this frequency is the one that contains the greatest error in the approximation with the Linear FEM. The results are shown in Tab. 8.

Table 8. Approximations of the second frequency with the GFEM in only one element – Different materials

Element	η_i	ω_2	Relative Error (%)
L_1	0.0165	1.71822379	9.39%
L_2	0.2651	1.57754874	0.43%

Again the application of the indicator obtained good results, since the addition of GFEM functions in the first element reduces the relative error for the second frequency is 9.39% and the value of the Friberg indicator in this element is smaller than the value of the calculated indicator for the second element. The addition of GFEM functions in the second element produces a reduction in the relative error to 0.43%.

When the two elements L_1 and L_2 are enriched at the same time with the functions of the GFEM, one has the approximation $\omega_2 =$ 1.57685678 and the relative error is 0.38%, however the size of the global system increases 66.66%. A very close relative error is obtained enriching just the element L_2 providing a very good solution with less computational effort.

4. CONCLUSIONS

The results using the Friberg error estimator were quite satisfactory, since it is possible to obtain approximations with very similar convergence rates and low computational cost, only observing which element decreases more the error of a given frequency. This procedure proved to be very efficient for simple examples however it can be extended for analysis of more complex engineering structures.

In practice the use of selective GFEM can generate interesting reductions in the size of the global system, once the indicator will point the regions of the mesh which provide greater convergence if they are enriched.

In addition to the reduction of the global system, discussed in cases 3.1 and 3.3, another application of the Friberg indicator in the GFEM is the use of it to make comparisons between different types of enrichment, which can provide an increase in the accuracy of the approximations, as showed in case 3.2.

5. ACKNOWLEDGEMENTS

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6. REFERENCES

Arndt M., 2009. *O Método dos Elementos Finitos Generalizado Aplicado à Análise de Vibrações Livres de Estruturas Reticuladas*. Doctoral Thesis, Universidade Federal do Paraná. Curitiba, Brazil.

Babuska, I., Banerjee U., 2012. *Stable Generalized Finite Element Method (SGFEM).* Comput. Methods Appl. Mech. Engrg. Vol. 201–204, pp. 91–111.

Duarte C.A., 2012. *A Generalized Finite Element Method for multiscale simulations*. University of Illinois at Urbana-Champaign 2122 Newmark Lab., MC-250 205 N. Mathews Av. Urbana, IL 61801.

Duarte H. V., 2003. *Estimador de Erro para a Formulação p do Método dos Elementos Finitos Aplicado ao Problema Fluido-Estrutura*, Doctoral Thesis, Universidade Estadual de Campinas. Campinas, Brazil.

Friberg P.O., 1986. *An error Indicator for the generalized eigenvalue problem using the hierarchical Finite Element Method*, International Journal for Numerical Methods in Engeneering, vol. 23, pp. 91-98.

Friberg P. O., Moller P., 1987. *An adaptive procedure for eigenvalue problems using the hierarchical Finite Element Method*, International Journal for Numerical Methods in Engeneering, vol. 24, pp. 319-335.

Gratsch T., Bathe K.J., 2005. *A posteriori error estimation techniques in practical finite element analysis.* Computers and Structures 83, 235-265.

Leite S.A., 2016. *Método de Elementos Espectrais para a equação da onda com coeficientes variáveis*. Doctoral Thesis, Universidade Federal do Paraná, Curitiba, Brazil.

Lins M.R., 2011. *Estimador de erro a posteriori baseado em recuperação do gradiente para o Método dos Elementos Finitos Generalizados*. Master’s thesis, Universidade São Paulo. São Carlos, Brazil.

Oden J.T., Reddy J.N., 1996. *An Introduction to the Mathematical Theory of Finite Elements*.[s.n.]. New York, USA.

Petroli T., 2016. *Condicionamento do problema de autovalores obtido do Método dos Elementos Finitos Generalizados na Dinâmica de Estruturas*. Master’s thesis, Universidade Federal do Paraná. Curitiba, Brazil.

Shang Y.H., 2014. *Análise Dinâmica Elastoplástica de Problemas da Mecânica de Sólidos via Métodos Enriquecidos de Elementos Finitos*, Doctoral Thesis, Pontifícia Universidade Católica do Paraná. Curitiba, Brazil.

Silva, J., 2015. *Análise de estimadores de erro a posteriori aplicados ao Método dos Elementos Finitos utilizando refino h-adaptativo*. . Master’s thesis, Universidade Federal do Paraná. Curitiba, Brazil.

Szabó B., Babuska I., 1991. *Finite element analysis*. John Wiley & Sons Inc. New York, USA.

Torii A.J., 2012. *Análise Dinâmica de Estruturas com o Método dos Elementos Finitos Generalizado*. Doctoral Thesis, Universidade Federal do Paraná, Curitiba, Brazil.

Torii A.J., Machado R.D., 2012. *Structural Dynamic Analysis For Time Response of Bars and Trusses Using the Generalized Finite Element Method*. Latin American Journal of Solids and Structures. Vol. 9, no.3. Rio de Janeiro, Brazil.

Weinhardt P.D., 2016. *Estudo da Estabilidade do Método dos Elementos Finitos Generalizados Aplicado à Análise Dinâmica*, Master’s Thesis. Universidade Federal do Paraná. Curitiba, Brazil.

Zienkiewicz O.C., Taylor R.L., 1977. *The finite element method*, McGraw-hill. London, England.

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