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# APPLICATION OF GALERKIN METHOD TO THE SOLUTION OF AERODYNAMIC MODELS BASED ON THE LIFTING-LINE THEORY

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***Abstract.*** *The lifting-line theory (LLT) provided the first reasonable prediction of the finite wings aerodynamic coefficients and it is still in use nowadays in the initial phases of the aircraft design. However, its fundamental equation has a singularity point, which led many authors to develop numerical procedures to overcome this problem. This paper proposes to solve the lifting-line theory equation via Galerkin method, which was compared to the classical solution via Fourier series. The results obtained for a rectangular and a trapezoidal wing showed a good agreement between the two techniques, with an absolute error inferior to* $33.9 \cdot 10^{-5}$ *for the circulation.*

***Keywords:*** *lifting-line theory, Galerkin method, computational aerodynamics*

## 1. INTRODUCTION

The design process of a new aircraft is highly iterative and multidisciplinary. It is usually divided in three different phases: conceptual, preliminary and detail design (Raymer, 2013). In this context, in which concerns to conceptual and preliminary design stages, it is necessary for the designer to have reasonable estimates of the aerodynamic coefficients of the entire aircraft to predict its performance and the requirement fulfillment (Houghton *et al.*, 2016). There are different tools capable to estimate the wing aerodynamic coefficients, such as computational fluid dynamics (CFD), wind-tunnel testing and analytical approach. One of these methods was developed independently about one hundred years ago by Frederick W. Lanchester and Ludwig Prandtl which is known as *Prandtl's Lifting-Line Theory* (LLT) or *Lanchester-Prandtl Theory* in some countries. Based on Biot-Savart law, this theory can be used to estimate wing load distributions and aerodynamic coefficients (Anderson, 2016).

The classical solution to Prandtl theory applies to single aerodynamic surfaces with no dihedral and sweep, what considerably restricts its applications in the dsign of modern aircrafts (Phillips and Snyder, 2000). To amplify the usage of this theory, many authors provided alternative solution methods for the fundamental equation. Von Kármán and Tsien (1945) presented a method to take into account the non-uniformity of the freestream. Anderson *et al.* (1980) presented a numerical iterative solution to Prandtl's lifting-line theory modified to predict poststall behavior for finite wings with and without leading-edge droop that showed reasonable agreement with experimental data. The coupled nonlinear system was discretized and solved by means of a global Newton method. According to the author, although there are more exact aerodynamic formulations, the computational effort for these methods is considerably greater. Rasmussen and Smith (1999) presented a method for solving the classical LLT for arbitrarily shaped wings which converges faster than collocation methods. Phillips and Snyder (2000) presented a solution strategy based on a fully three-dimensional vortex lifting law which is applicable for lifting surfaces with arbitrary camber, sweep and dihedral. Numerical solutions has been proposed by Ziller (1940) and Multhopp (1950), whose implementations have advantages for being robust, specially the first one for using the Rayleigh-Ritz method. Recently, Gallay *et al.* (2014), Gallay and Laurendeau (2015) and Nguyen *et al.* (2017) presented works based on LLT, demonstrating that there are studies in this field nowadays.

In this context, the main goal of this paper is to apply Galerkin method to solve the classical lifting-line theory fundamental equation and compare its results with the classical Fourier series solution. Galerkin method is an integral method, which according to Anderson (1964) has the advantage when the effects of flaps and surface controls are considered.

## 2. LIFTING-LINE MODEL

Firstly, in this section, the general concepts involving the lifting-line equation and its classical solution via Fourier series are reviewed to provide a more self-contained discussion. Lifting-line models are built considering a vortex system (Figure 1) in which the vortex strength at a wing station $y$ is given by $\Gamma(y)$. The circulation distribution ($\Gamma$) induces the downwash $w(y)$, which changes the local angle of attack. The local angle of attack variation is given by:

$$\Delta\alpha = \tan^{-1}(w(y)/U_\infty) \approx w(y)/U_\infty \tag{1}$$

The downwash at a station $y$ can be written as a sum of the circulation contribution of each wing station, i.e.:

$$w(y) = \frac{-1}{4\pi}\int_{-b/2}^{b/2} \frac{1}{y-t}\frac{d\Gamma}{dt}\,dt \tag{2}$$

where $t, y \in [-b/2, b/2]$ are wing stations.

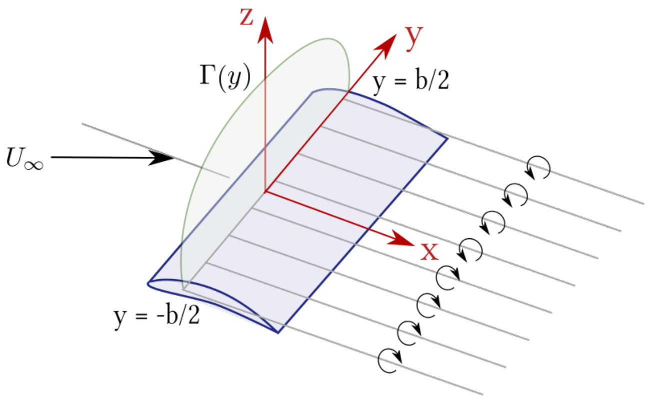


Figure 1. Distribution of circulation $\Gamma$ along the span of a rectangular wing

The lift coefficient at a station $y$ can be written as:

$$c_l(y) = \pi\rho U_\infty = a_0(\alpha - \alpha_{L=0}) \tag{3}$$

where $\rho$ is the local air density, $U_\infty$ is the freestream velocity, $a_0$ is the slope of $C_l$ vs. $\alpha$ curve of the airfoil, $\alpha$ is the wing local angle of attack and $\alpha_{L=0}$ is the zero-lift angle of attack. The lifting-line presented herein is the linear model, so there is no iterative process to recompute the circulation distribution after the computation of the local angle of attack.

Combining the equations (1), (2) and (3), it is obtained the well-known *fundamental equation of Prandtl's lifting-line theory* (Anderson, 2016):

$$\alpha(y) = \frac{2}{a_0(y)U_\infty c(y)}\Gamma(y) + \alpha_{L=0}(y) - \frac{1}{4\pi U_\infty}\int_{-b/2}^{b/2}\frac{1}{y-t}\frac{d\Gamma(t)}{dt}\,dt, \quad y,t \in [-b/2, b/2] \tag{4}$$

The above equation is an integro-differential equation with a Cauchy kernel in the unknown $\Gamma(y)$ (Mandal and Chakrabarti, 2016). It is important to highlight the singular kernel at $y = t$ of the integral, which requires special numerical treatment to overcome it (Kythe and Puri, 2011). The finite wing lift coefficient, $C_L$, and the induced drag coefficient, $C_{D_i}$, can be computed as:

$$C_L = \frac{2}{U_\infty S}\int_{-b/2}^{b/2}\Gamma(t)\,dt \quad \text{and} \quad C_{D_i} = \frac{2}{U_\infty S}\int_{-b/2}^{b/2}\Gamma(t)\alpha_i(t)\,dt \tag{5}$$

The classical generalized solution of Eq. (4) is given by a Fourier series:

$$\Gamma(\theta) = 2bU_\infty \sum_{i=1}^{N} A_i \sin(i\theta) \tag{6}$$

where $\theta$ is an auxiliary variable so that $y = -(b/2)cos(\theta)$.

Substituting the Eq. (6) in the Eq. (4), it is possible to compute the angle of attack as:

$$\alpha(\theta) = \frac{2b}{\pi c(\theta)}\sum_{i=1}^{N} A_i sin(i\theta) + \alpha_{L=0}(\theta) + \sum_{i=1}^{N} iA_i\frac{sin(i\theta)}{sin(\theta)} \tag{7}$$

It is known that Eq. (7) is actually a set of algebraic equations with N unknowns, namely, $A_1$,...,$A_N$. Solving this linear system, the lift and induced drag coefficients can be computed as follows:

$$C_L = A_1 \pi A\!\!R \quad \text{and} \quad C_{D_i} = \frac{C_L^2}{\pi A\!\!R}(1+\delta) \tag{8}$$

where $A\!\!R$ is the wing taper ratio and $\delta$ is given by:

$$\delta = \sum_{i=2}^{N} i \left(\frac{A_i}{A_1}\right)^2 \tag{9}$$

## 3. NUMERICAL PROCEDURE

At this section, the numerical procedure based on Galerkin method formulation is presented. This method consists in using a linear combination of functions to approximate the unknown field, $\Gamma(y)$. The application of this approximation to the weak form leads to a linear system of equations, where the unknowns are the multipliers of each functions of the function basis. For the development of the numerical procedure, it is convenient to rewrite Eq. (4) in terms of new functions $d(y)$, $g(y)$ and a constant $\eta_1$ as follows:

$$g(y)\,\Gamma(y) - \eta_1 \int_{-b/2}^{b/2} \frac{1}{y-t}\frac{d\Gamma}{dt}\,dt - d(y) = 0 \tag{10}$$

where:

$$g(y) = \frac{2}{a_0(y)c(y)U_\infty}; \quad \eta_1 = \frac{1}{4\pi U_\infty}; \quad \text{and} \quad d(y) = \alpha(y) - \alpha_{L=0}(y) \tag{11}$$

The weak form of Eq. 10 is written as:

$$\int_{-b/2}^{b/2} \delta\Gamma(y)\left[g(y)\Gamma(y) - \eta_1 \int_{-b/2}^{b/2} \frac{1}{y-t}\frac{d\Gamma}{dt}\,dt - d(y)\right] dy = 0 \tag{12}$$

where $\Gamma(y)$ and $\delta\Gamma(y)$ are the trial and test functions, respectively. It must be reminded that $\Gamma(y)$ is enough continuous and smooth on essential boundary conditions. These functions are given by:

$$\Gamma(y) = \sum_{i=1}^{N} \gamma_i \phi_i(y) \qquad \text{and} \qquad \delta\Gamma(y) = \sum_{j=1}^{N} \beta_j \phi_j(y). \tag{13}$$

where $\phi_i(y)$ and $\phi_j(y)$ are basis functions. It is important to remember that $\Gamma(\pm b/2) = 0$. It's known that $\phi_i(y)$ is a hat function. The function $\phi_i(y)$ can be written as:

$$\phi_i(y) = \begin{cases} \dfrac{y - y_{i-1}}{y_i - y_{i-1}}, & \text{if } y \in [y_{i-1}, y_i] \\[2ex] \dfrac{y - y_{i+1}}{y_i - y_{i+1}}, & \text{if } y \in [y_i, y_{i+1}] \\[2ex] 0, & \text{if } y < y_{i-1} \cup y > y_{i+1} \end{cases} \tag{14}$$

and its derivative is a constant. This result simplifies the numerical treatment of Eq. (12). Adopting a basis function described above, the values of the linear combination coefficients, $\gamma_i$, have the physical meaning of being the circulation at station $y = y_i$.

The equations (13) are substituted in (12) to make the solution independent of $\beta_j$, because the weight function is also sufficiently smooth and must vanish on essential boundaries conditions (Fish and Belytschko, 2007). It results in a linear system where the vector of the circulations in each node ($\gamma_i$) is the unknown. The circulations $\alpha_i$ are the solution of the linear system:

$$\boldsymbol{\gamma} = \mathbf{A}^{-1}\mathbf{f} = (\mathbf{A_1} + \mathbf{A_2})^{-1}\mathbf{f} \tag{15}$$

where:

$$(\mathbf{A_1})_{ij} = \int_{-b/2}^{b/2} g(y)\phi_j(y)\phi_i(y)\,dy, \tag{16}$$

$$(\mathbf{A_2})_{ij} = -\eta_1 \int_{-b/2}^{b/2} \phi_j(y) \int_{-b/2}^{b/2} \frac{1}{y-t} \frac{d\phi_i(t)}{dt}\, dt\, dy, \text{ and} \tag{17}$$

$$\mathbf{f_i}(y) = \int_{-b/2}^{b/2} d(y)\, \phi_j(y)\, dy \tag{18}$$

In each element, $\mathbf{A_2}$ undergoes a change of its limits of integration, which is shown below:

$$(\mathbf{A_2})_{ij} = -\eta_1 \int_{y_{j-1}}^{y_{j+1}} \phi_j(y) \int_{y_{i-1}}^{y_{i+1}} \frac{1}{y-t} \frac{d\phi_i(t)}{dt}\, dt\, dy \tag{19}$$

Herein, without loss of generality, the term $d(y)$ is considered as constant $\eta_2$, i.e., no geometric twist has been taken into account.

The Gauss-Legendre quadrature of fourth degree was used to evaluate the integrals of the problem. As an example, $(\mathbf{A_2})_{ij}$ is given by the sum of $(\mathbf{A_2^1})_{ij}$ and $(\mathbf{A_2^2})_{ij}$, and becomes:

$$(\mathbf{A_2^1})_{ij} = -\eta_1 \left(\frac{y_j - y_{j-1}}{2}\right) \left[\sum_{k=1}^{N} \left(\frac{1+\xi^{(k)}}{2}\right) \frac{q_1}{h_1} w^{(k)} - \sum_{k=1}^{N} \left(\frac{1+\xi^{(k)}}{2}\right) \frac{q_2}{h_2} w^{(k)}\right] \tag{20}$$

$$(\mathbf{A_2^2})_{ij} = -\eta_1 \left(\frac{y_{j+1} - y_j}{2}\right) \left[\sum_{k=1}^{N} \left(\frac{1+\xi^{(k)}}{2}\right) \frac{q_1}{h_1} w^{(k)} - \sum_{k=1}^{N} \left(\frac{1+\xi^{(k)}}{2}\right) \frac{q_2}{h_2} w^{(k)}\right] \tag{21}$$

where $\bar{y}$, $h$ and $q$ are, respectively:

$$\bar{y}_{1,2}^{(k)} = \begin{cases} (y_j - y_{j-1})\frac{\xi^{(k)}+1}{2} + y_{j-1}, & \text{if } y \in [y_{j-1}, y_j] \\ \\ (y_{j+1} - y_j)\frac{\xi^{(k)}+1}{2} + y_j, & \text{if } y \in [y_j, y_{j+1}] \end{cases} \tag{22}$$

$$h_{1,2} = \begin{cases} y_i - y_{i-1} \\ \\ y_{i+1} - y_i \end{cases}, \text{ and} \tag{23}$$

$$q_{1,2}^{(k)} = \begin{cases} \ln\left|\frac{\bar{y}_1^{(k)} - y_i}{\bar{y}_1^{(k)} - y_{i-1}}\right| \\ \\ \ln\left|\frac{\bar{y}_2^{(k)} - y_{i+1}}{\bar{y}_2^{(k)} - y_i}\right| \end{cases} \tag{24}$$

It must be reminded that the values of $\xi^{(k)}$ and $w^{(k)}$ are the integration points and weights of the Legendre polynomials, respectively.

## 4. RESULTS AND DISCUSSION

The proposed numerical strategy has been applied for two straight wings: (A) rectangular and (B) tapered wing. In this section, the comparison between the classical Fourier series solution and the proposed method were presented. The profile data required for the computations, viz. the angle of zero lift ($\alpha_{L=0}$) and the slope of the polar $C_\ell$ vs. $\alpha$ ($a_0$), were obtained using XFoil, considering the freestream velocity to compute the Reynolds number. It is important to highlight that for the considered airfoil, the influence of the Reynolds number at low angles of attack is negligible, as can be seen in Abbott and Von Doenhoff (1959). For the lifting-line computations, the same Reynolds number was kept constant along the spanwise, even for the tapered wing (Example B). It was computed an angle of attack of zero lift equal to $-1.213°$ and $a_0 = 6.1311\ rad^{-1}$.

### 4.1 Example A - Rectangular wing

In this first example, the developed tool was applied for a rectangular wing with an aspect ratio of $AR = 7.42$, a wing area of $S = 16.3\ m^2$, and a NACA 23015 profile within a freestream of $U_\infty = 60\ m/s$ at sea level ($\rho = 1.225\ kg/m^3$). The wing has no geometric or aerodynamic torsion.

Table 1. Lift coefficients for several angle of attacks predicted by classical Fourier series solution and via Galerkin method for a rectangular wing.

| | Lift coefficient | | |
|---|---|---|---|
| $\alpha[°]$ | Galerkin | Fourier | Absolute error $(10^{-5})$ |
| -4 | -0.2272377 | -0.2272159 | 2.18 |
| -2 | -0.0641679 | -0.0641618 | 0.615 |
| 0 | 0.0989018 | 0.0988923 | 0.949 |
| 2 | 0.2619716 | 0.2619465 | 2.51 |
| 4 | 0.4250414 | 0.4250006 | 4.08 |
| 6 | 0.5881112 | 0.5880548 | 5.64 |

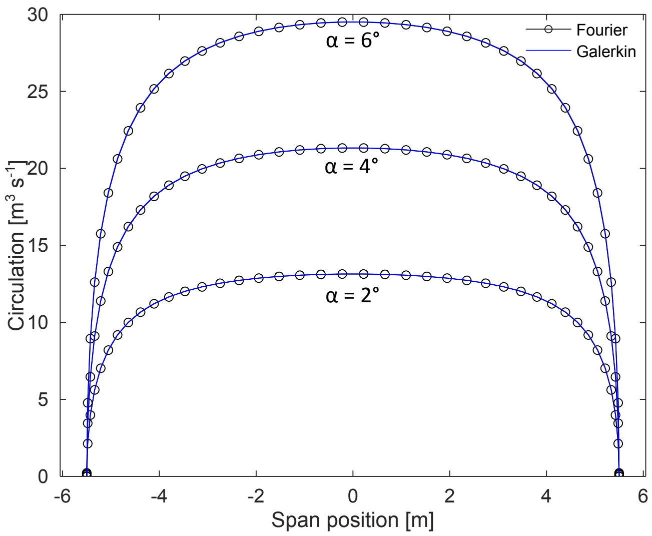


Figure 2. Rectangular wing span circulation distribution for $\alpha = 2, 4$ e $6°$. Comparison between Classical LLT solution and Galerkin method (cosinusoidal mesh).

The wing was discretized using 40 terms for the Fourier series and 40 nodes for the Galerkin method. The positions of the nodes in the Galerkin method were determined according to a cosinusoidal distribution. The lift coefficients were computed for both methods and their values are shown in Table 1. To illustrate the equivalence between the methods, the circulation distribution can be seen in Figure 2 for $\alpha = 2, 4$ e $6°$.

It is possible to notice that, in both cases, the agreement between the methods is remarkable, showing that this formulation is adequate for the analysis of straight rectangular wings. The absolute error was kept smaller than $6 \cdot 10^{-5}$ and the shape of the circulation distribution is the same when compared to the analysis via Fourier series for all the angles of attack in this range.

### 4.2 Example B - Tapered wing

In this example, the tool was applied for the same wing data with a taper ratio $\lambda = 0.5$ in the same freestream condition. The comparison between the two solution strategies (Fourier and Galerkin) was shown in Table 2. The absolute error was evaluated and for the linear region of the $C_L$ vs. $\alpha$ curve it was not greater than $31.9 \cdot 10^{-5}$, showing satisfactory agreement of the proposed solution method with the classical one. It can be seen for this case that the absolute error is higher than for a rectangular wing, but still sufficiently small.

The circulation along the span is shown in Figure 3 and, as in Example A, the results obtained via Galerkin method adopted herein are in agreement with the ones obtained using the Fourier series solution, demonstrating the applicability of the method for the case of tapered wings.

Finally, a convergence analysis was performed for $\alpha = 4°$ to investigate the influence of the discretization – number of series terms for Fourier approximation and number of nodes of the Galerkin method – in the computed lift coefficient (Figure 4). The Galerkin method was applied for both uniform and cosinusoidal meshes. It can be noticed that Fourier series method converges faster than both uniform and cosinusoidal discretizations, although the convergence of the uniform case took more than ten times the number of nodes required for the cosinusoidal discretization. The computation of

Table 2. Lift coefficients for several angle of attacks predicted by classical Fourier series solution and via Galerkin method for a tapered wing.

| | Lift coefficient | | |
|---|---|---|---|
| $\alpha[°]$ | Galerkin | Fourier | Absolute error ($10^{-5}$) |
| -4 | -0.2332850 | -0.2331618 | 6.67 |
| -2 | -0.0658756 | -0.0658408 | 3.48 |
| 0 | 0.1015338 | 0.1014802 | 5.36 |
| 2 | 0.2689432 | 0.2688012 | 14.2 |
| 4 | 0.4363526 | 0.4361223 | 23.0 |
| 6 | 0.6037621 | 0.6034433 | 31.9 |

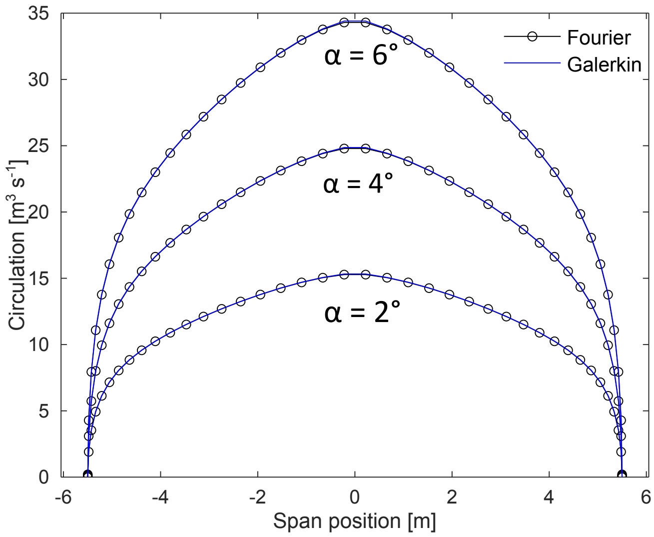


Figure 3. Tapered wing span circulation distribution for $\alpha = 2$, 4 e $6°$. Comparison between Classical LLT solution and Galerkin method (cosinusoidal mesh).

the $\mathbf{A}_2$ matrix is the most time-consuming part of the algorithm, since it is a full matrix. Its computation takes $87.8\%$ of the total time of the Galerkin method implementation, making it more time-consuming than Fourier's series solution.

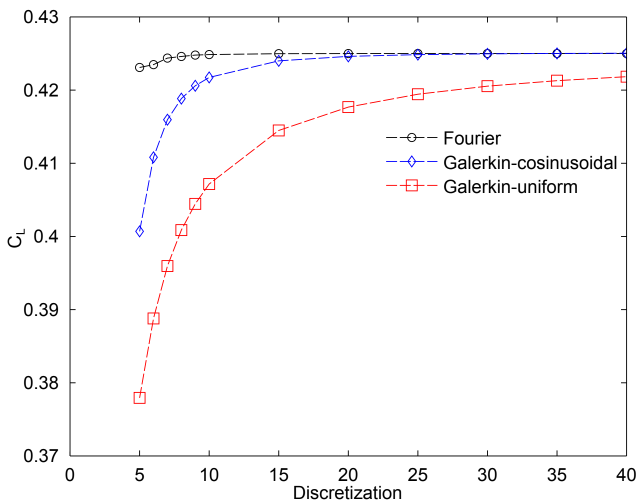


Figure 4. Influence of the discretization and meshing strategy in Galerkin solution. The results are shown for the rectangular wing at $\alpha = 4°$.

## 5. CONCLUSIONS

The presented numerical strategy demonstrated to be in good agreement with the classical solution for the lifting-line problem via Fourier series for both rectangular and trapezoidal wing. The convergence analysis showed that mesh refinement in the wing tips has a major influence on the convergence of Galerkin method. It also showed that Fourier's classical solution converges faster than the proposed method. However, Galerkin method can be suitable when it is necessary to modify the wing properties along the span. Thereby, the range of applications of this method is wider when compared to other ones.

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## 7. REFERENCES

Abbott, I.H. and Von Doenhoff, A.E., 1959. *Theory of Wing Sections: Including a Summary of Airfoil data*, Vol. 11. Dover Publications, INC., New York.

Anderson, J.D., 2016. *Fundamentals of Aerodynamics*. McGraw-Hill Education.

Anderson, J.D., Corda, S. and Van Wie, D.M., 1980. "Numerical lifting line theory applied to drooped leading-edge wings below and above stall". *Journal of Aircraft*, Vol. 17, No. 12, pp. 898–904.

Anderson, R.C., 1964. "Application of the Galerkin Method to the Prandtl Lifting Line Equation." Vol. 1, No. 3, pp. 126–128.

Fish, J. and Belytschko, T., 2007. *A First Course in Finite Elements*. Wiley.

Gallay, S. and Laurendeau, E., 2015. "Nonlinear Generalized Lifting-Line Coupling Algorithms for Pre/Poststall Flows". *AIAA Journal*, Vol. 53, No. 7, pp. 1784–1792.

Gallay, S., Ghasemi, S. and Laurendeau, E., 2014. "Sweep effects on non-linear lifting line Theory near stall". *AIAA SciTech, National Harbor, Maryland, 52nd Aerospace Sciences Meeting*, , No. January, pp. 1–19.

Houghton, E., Carpenter, P., Collicott, S. and Valentine, D., 2016. *Aerodynamics for Engineering Students*. Elsevier Science.

Kythe, P. and Puri, P., 2011. *Computational Methods for Linear Integral Equations*. Birkhäuser Boston.

Mandal, B. and Chakrabarti, A., 2016. *Applied Singular Integral Equations*. CRC Press.

Multhopp, H., 1950. "Methods for Calculating the Lift Distribution of Wings (Subsonic Lifting-Surface Theory)." *Aeronautical Research Council*, Vol. 2884, No. 2884.

Nguyen, N.T., Ting, E. and Chaparro, D., 2017. "Nonlinear Large Deflection Theory with Modified Aeroelastic Lifting Line Aerodynamics for a High Aspect Ratio Flexible Wing". *35th AIAA Applied Aerodynamics Conference*, , No. June, pp. 1–26.

Phillips, W.F. and Snyder, D.O., 2000. "Modern Adaptation of Prandtl's Classic Lifting-Line Theory". *Journal of Aircraft*, Vol. 37, No. 4, pp. 662–670.

Rasmussen, M.L. and Smith, D.E., 1999. "Lifting-line theory for arbitrarily shaped wings". *Journal of Aircraft*, Vol. 36, No. 2, pp. 340–348.

Raymer, D., 2013. *Aircraft Design - RDS-Student: A Conceptual Approach*. American Institute of Aeronautics & Astronautics.

Von Kármán, T. and Tsien, H.S., 1945. "Lifting-Line Theory for a Wing in Non-uniform Flow". *Collected Works of Hsue-Shen Tsien (1938-1956)*, , No. 2012, pp. 286–296.

Ziller, F., 1940. "Beitrag zur theorie des tragflügels von endlicher spannweite". *Archive of Applied Mechanics*, Vol. 11, No. 4, pp. 239–259.

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