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COMPARISON OF EIGENPROBLEM SOLUTION APPROACHES FOR DIRICHLET CONDITIONS IN IRREGULAR DOMAINS VIA INTEGRAL TRANSFORMS

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Abstract. *This paper is focused on the solution of eigenvalue problems for irregular-shaped domains using the Generalized Integral Transform Technique (GITT). Two-dimensional Helmholtz problems in cartesian coordinates with Dirichlet boundary conditions are used as test-cases with a semi-circular region with angles $\phi = 90°$ and $\phi = 180°$. The solution of the eigenproblem is carried out by using an auxiliary problem constructed from simpler 1D eigenfunctions as a basis for expanding the sough solution, and two-different approaches are comparative analyzed. The first is based on using an auxiliary problem defined within the same region as the original problem (the coincident domain approach – CDA), while the second relies on using an auxiliary problem defined in a fictitious domain, obtained from expanding the original domain to a larger regular domain (the fictitious domain approach – FDA). The solutions are verified by comparing the calculated results of the semi-circular domain with the exact analytical solution obtained by employing cylindrical coordinates.*

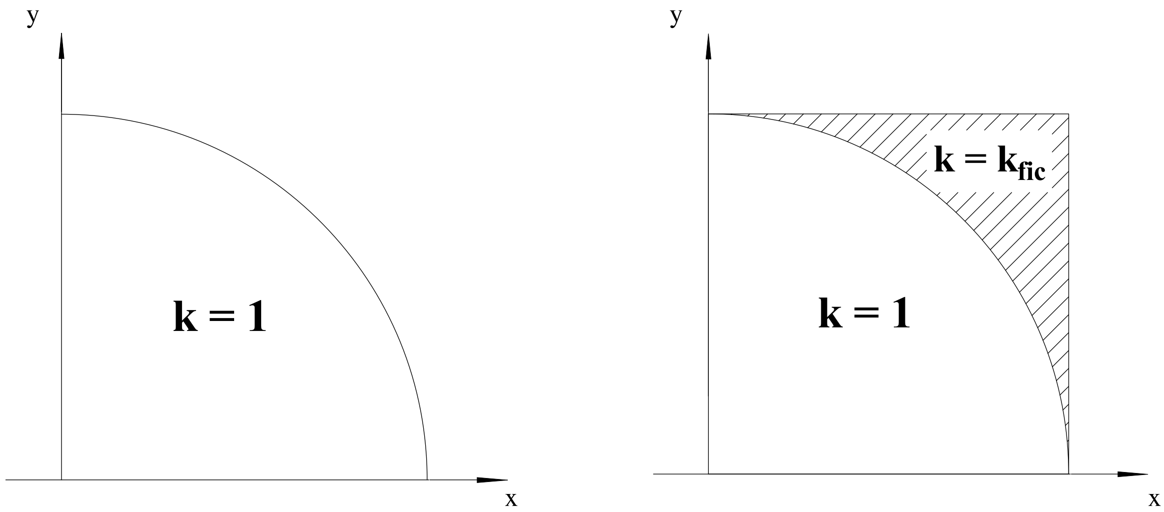
Keywords: *Irregular Domains, GITT, Eigenproblem, Symbolical Computation*

NOMENCLATURE

A, B, C, M	integral coefficients
k	coefficient
w	weighting function
d, G	know functions
$\mathcal{A}$	area
$\mathcal{S}$	surface
$\mathcal{V}$	volume
x, y	spatial coordinates
X, Y	auxiliary eigenfunctions
Greek Symbols	
β, ω	auxiliary eigenvalues
δ	Kronecker delta function
μ	eigenvalues
Ψ	eigenfunctions
Ω	basis of auxiliary eigenfunctions
ϕ	angle
Subscripts	
i, j, k, l, m, n	summation indices
c	circle
fic	fictitious
max	maximum
Overscripts	
–	transformed function

1. INTRODUCTION

Previously to the interpretation of a studied problem, many considerations need to be taken into account primarily and the choice of the working domain is a concerning subject to consider. There is no denying that an abundant number of research papers in the literature deals with different problems in traditional regular geometries, however most engineering applications are actually formulated for irregular geometries. Based on that, when irregular domains are treated, a numerical methodology is required to obtain the solution of eigenproblems. Due to the hybrid analytical-numerical nature of the Generalized Integral Transform Technique (GITT) (Cotta, 1993), this method seems a convenient choice to the



(a) CDA (b) FDA

Figure 1: Representation of both approaches in semi-circular domain.

development of the eigenvalue problem applied to irregular-shaped domains.

The use of the GITT method is a fairly common methodology applied to regular geometries, as different applications of the method can be cited, for example, studies presenting the GITT for thermally developing flows in rectangular ducts (Chalhub and Sphaier, 2010; Aparecido and Cotta, 1990), for forced convection analysis in rectangular cross-section (Lindquist and Aparecido, 1999) and slip-flow in microchannels (Yu and Ameel, 2001). However, even sparse, some investigations into irregular domains can also be presented, as in fully developed flows (Barbuto and Cotta, 1997; Pérez Guerrero *et al.*, 2000), fluid flow in irregularly-shaped microchannels (Pinheiro and Sphaier, 2016) or in specific studies into eigenproblems in irregular-shaped geometries (Sphaier and Cotta, 2000).

The steps for the GITT approach requires the solution of the Sturm-Liouville eigenvalue problem, which determines the set of eigenfunctions, related to corresponding eigenvalues. In the literature, the eigenproblems can incorporate primarily importance in specific works, such as the one presented by Cotta *et al.* (2016), in which nonlinear convection-diffusion problems yield also nonlinear eigenvalue problem solved by means of the GITT method. Early works involving eigenploblems can be exemplified by the work proposed by Tai and Shaw (1974), where the two or three-dimensional Helmholtz-based eigenvalue problem is numerically solved in arbitrarily-shaped domains.

As many engineering problems are indeed applied for irregular or more complex geometries, this study presents two different strategies to solve eigenproblems in irregularly-shaped domains, in which Dirichlet boundary conditions are considered. One approach is based on the accordance of the actual geometry and the domain related to the mathematical formulation. This approach is named the Coincident Domain Approach (CDA). Regarding the second approach, the Fictitious Domain Approach (FDA) requires the definition of a regular domain outside the irregular geometry, characterized by a fictitious coefficient k_{fic}. Both approaches (figure 1) are solved via GITT with the aid of the Mathematica Platform (Wolfram, 2003) and a convergence analysis is employed with a semi-circular region.

2. MATHEMATICAL FORMULATION

The Eigenvalue problem is represented by the Sturm-Liouville Equation, in which the eigenvalues are illustrated by μ and the corresponding eigenfunctions are characterized by Ψ. The generalized eigenproblem is established for an arbitrary domain $\mathcal{V}$ with delimited surface $\mathcal{S}$, as can be seen in the following equations:

$$\nabla \cdot (k(\boldsymbol{x})\, \nabla \Psi) \,+\, (\mu^2\, w(\boldsymbol{x}) - d(\boldsymbol{x}))\, \Psi \,=\, 0, \qquad \boldsymbol{x} \in \mathcal{V}, \tag{1}$$

$$k(\boldsymbol{x})\, (\nabla \Psi \cdot \boldsymbol{n}) + \gamma(\boldsymbol{x})\, \Psi = 0, \qquad \boldsymbol{x} \in \mathcal{S}, \tag{2}$$

where $w(x)$ is considered a weighting function and $d(x)$ is known function. In order to apply the GITT approach, the auxiliary transform-inverse pair is described:

$$\Psi_n \,=\, \sum_{i=1}^{\infty} \Omega_i(\boldsymbol{x})\, \bar{\Psi}_i^n, \tag{3}$$

$$\bar{\Psi}_i^n \,=\, \int_{\mathcal{V}^*} w^*(\boldsymbol{x})\, \Psi_n(\boldsymbol{x})\, \Omega_i(\boldsymbol{x})\, \mathrm{d}\mathcal{V} \tag{4}$$

The auxiliary eigenfunctions are orthogonal, so:

$$\int_{\mathcal{V}^*} w^*(\boldsymbol{x})\,\Omega_i(\boldsymbol{x})\,\Omega_j(\boldsymbol{x})\,\mathrm{d}\mathcal{V} = \delta_{i,j}, \tag{5}$$

in which $\delta_{i,j}$ is the Kronecker delta function and $w^*(\boldsymbol{x})$ is the weighting function, now associated with functions Ω.

The procedure to transform the original eigenproblem requires the multiplication of the differential equation (1) by the auxiliary eigenfunctions Ω, integrating within the considered domain $\mathcal{V}_T$, delimited by surface $\mathcal{S}_T$. The domain can be related to $\mathcal{V}$ or $\mathcal{V}^*$, depending only on the strategy applied (Coincident Domain Approach or Fictitious Domain Approach).

$$\int_{\mathcal{V}_T} \nabla\cdot(k\,\nabla\Psi)\,\Omega_i\,\mathrm{d}\mathcal{V} + \mu^2 \int_{\mathcal{V}_T} w\,\Psi\,\Omega_i\,\mathrm{d}\mathcal{V} - \int_{\mathcal{V}_T} d\,\Psi\,\Omega_i\,\mathrm{d}\mathcal{V} = 0 \tag{6}$$

In the first integral, the second Green formula is used, which gives:

$$\int_{\mathcal{V}_T} \nabla\cdot(k\,\nabla\Omega_i)\,\Psi\,\mathrm{d}\mathcal{V} + \int_{\mathcal{S}_T} k\,(\Omega_i\,\nabla\Psi - \Psi\,\nabla\Omega_i).\boldsymbol{n}\,\mathrm{d}\mathcal{A} + \mu^2 \int_{\mathcal{V}_T} w\,\Psi\,\Omega_i\,\mathrm{d}\mathcal{V} - \int_{\mathcal{V}_T} d\,\Psi\,\Omega_i\,\mathrm{d}\mathcal{V} = 0 \tag{7}$$

And the first Green formula gives:

$$-\int_{\mathcal{V}_T} (\nabla\Omega_i)\cdot(k\,\nabla\Psi)\,\mathrm{d}\mathcal{V} + \int_{\mathcal{S}_T} [k\,(\Omega_i\,\nabla\Psi)]\cdot\boldsymbol{n}\,\mathrm{d}\mathcal{A} + \mu^2 \int_{\mathcal{V}_T} w\,\Psi\,\Omega_i\,\mathrm{d}\mathcal{V} - \int_{\mathcal{V}_T} d\,\Psi\,\Omega_i\,\mathrm{d}\mathcal{V} = 0 \tag{8}$$

Both first and second Green identities can be applied for the considered problem, however when discontinuous regions are in place, the first Green formula is recommended. The same apply for the FDA, because the first Green formula results in simpler coefficients to be calculated.

Applying the Dirichlet boundary condition on the irregular domain surface, one can infer that:

$$\Psi = 0 \qquad \text{and} \qquad \Omega_i = 0 \quad \text{in} \quad \boldsymbol{x}\in\mathcal{S}_T \tag{9}$$

which reduces the surface integrals of equations (7) and (8) to zero.

After substituting the integrals for coefficients A, B, C and D, the final transformed eigenproblem for both first and second Green identities are given by:

$$\sum_{j=1}^{\infty}(-D_{i,j} + \mu^2\,B_{i,j} - C_{i,j})\,\bar{\Psi}_j = 0 \tag{10}$$

$$\sum_{j=1}^{\infty}(A_{i,j} + \mu^2\,B_{i,j} - C_{i,j})\,\bar{\Psi}_j = 0 \tag{11}$$

where A, B, C and D are specifically:

$$A_{i,j} = \int_{\mathcal{V}_T} \nabla\cdot(k\,\nabla\Omega_j)\,\Omega_i\,\mathrm{d}\mathcal{V} \tag{12}$$

$$B_{i,j} = \int_{\mathcal{V}_T} w\,\Omega_j\,\Omega_i\,\mathrm{d}\mathcal{V} \tag{13}$$

$$C_{i,j} = \int_{\mathcal{V}_T} d\,\Omega_j\,\Omega_i\,\mathrm{d}\mathcal{V} \tag{14}$$

$$D_{i,j} = \int_{\mathcal{V}_T} (\nabla\Omega_i)\cdot(k\,\nabla\Omega_j)\,\mathrm{d}\mathcal{V}, \tag{15}$$

Another way to write the proposed solution can be seen below:

$$\sum_{j=1}^{\infty}(M_{i,j} + \mu^2\,B_{i,j})\,\bar{\Psi}_j = 0 \tag{16}$$

where the integral coefficient M is the summation of both A and C or D and C, as observed:

$$M_{i,j} = A_{i,j} - C_{i,j} \tag{17}$$

$$M_{i,j} = -D_{i,j} - C_{i,j} \tag{18}$$

The system is then truncation to a finite number of terms, giving the eigenvalues (μ) and the eigenvectors ($\bar{\Psi}$).

Regarding the considering domain, when discussing the CDA, it requires that the domain and the actual considered geometry be coincident, in which $\mathcal{S} = \mathcal{S}^* = \mathcal{S}_T$ and $\mathcal{V} = \mathcal{V}^* = \mathcal{V}_T$. However, in the FDA, although $\mathcal{V}^* = \mathcal{V}_T$, the problem is extended to fictitious domain, which included the irregular domain, in other words $\mathcal{V} \subset \mathcal{V}_T$. For the considered test-case, the extended domain is considered a rectangular geometry, which includes the semi-circular region.

3. TEST-CASE

For the considered test-cases with both strategies, the differential equation is based on a two-dimensional Helmholtz eigenproblem, as can be seen:

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \mu^2 \, \Psi \, = \, 0, \qquad 0 \le y \le y_1(x), \qquad 0 \le x \le 1, \tag{19}$$

$$\Psi(x, 0) \, = \, 0, \qquad 0 \le x \le 1, \tag{20}$$

$$\Psi(0, y) \, = \, 0, \qquad 0 \le y \le y(0), \tag{21}$$

$$\Psi(0, y(x)) \, = \, 0, \qquad y = y(x), \qquad 0 \le x \le 1, \tag{22}$$

where auxiliary eigenfunctions are determined by:

$$\Omega_i(x, y) = X_n(x) \, Y_k(y), \tag{23}$$

in which the solutions for X_n and Y_k are obtained by the following one-dimensional Helmholtz problems:

$$\frac{\partial^2 Y_k}{\partial y^2} + \omega_k^2 \, Y_k \, = \, 0, \tag{24}$$

$$\frac{\mathrm{d}^2 X_n}{\mathrm{d}x^2} + \beta_n^2 \, X_n \, = \, 0, \tag{25}$$

For both CDA and FDA schemes, the boundary conditions for the 1D eigenproblems are:

$$X(0) \, = \, 0, \qquad X(1) \, = \, 0, \qquad Y(y = 0) \, = \, 0, \tag{26}$$

3.1 CDA

The CDA is commonly seen in many problems presented in the literature. It requires that the considered region coincides with the domain applied in the mathematical formulation. For that, the boundary condition in the irregular region is represented by:

$$Y = 0, \quad \text{at} \quad y = y_c(x) \tag{27}$$

The function y_c is the equation for the semi-circular region, which is:

$$y_c(x) = \sqrt{1 - x^2} \tag{28}$$

Considering the known function d equals zero and integral coefficients in two-dimensional cartesian coordinates, A and B are characterized by:

$$A_{i,j} = \int_0^1 \int_0^{y_c(x)} \left[X_n \, \frac{\partial^2 Y_k}{\partial x^2} + 2 \, \frac{\partial X_n}{\partial x} \, \frac{\partial Y_k}{\partial x} - (\beta_n^2 + \omega_k^2) \, X_n \, Y_k \right] \, X_m \, Y_l \, \mathrm{d}y \, \mathrm{d}x \tag{29}$$

$$B_{i,j} = \int_0^1 \int_0^{y_c(x)} X_m \, Y_l \, X_n \, Y_k \, \mathrm{d}y \, \mathrm{d}x \tag{30}$$

The integral coefficient B is the same for both CDA and FDA. As for the integral coefficient A, it is acquired by the routine **NIntegrate** from the *Mathematica* platform.

3.2 FDA

The proposed FDA scheme requires the definition of an extended domain which contains the irregular-shaped region but doesn't match it. Because of that, the Dirichlet boundary condition is to be considered:

$$Y = 0, \quad \text{at} \quad y = 1 \tag{31}$$

which implies that the extended surface is a rectangular one. This extended domain is characterized by a coefficient k_{fic}, so the original coefficient k can be expressed by:

$$k = 1, \quad \text{if} \quad y \le y_c(x) \qquad \text{or} \qquad k = k_{fic}, \quad \text{if} \quad y > y_c(x) \tag{32}$$

The integral coefficient D is then expressed by:

$$D_{i,j} = \int_0^1 \int_0^1 k \left[X_n\, X_m\, \frac{\partial Y_l}{\partial y}\, \frac{\partial Y_k}{\partial y} + \left(Y_l\, \frac{\partial X_n}{\partial x} + X_n\, \frac{\partial Y_l}{\partial x} \right) \left(Y_k\, \frac{\partial X_m}{\partial x} + X_m\, \frac{\partial Y_k}{\partial x} \right) \right] \mathrm{d}y\, \mathrm{d}x, \tag{33}$$

which can be rewritten to avoid too many numerical integrations:

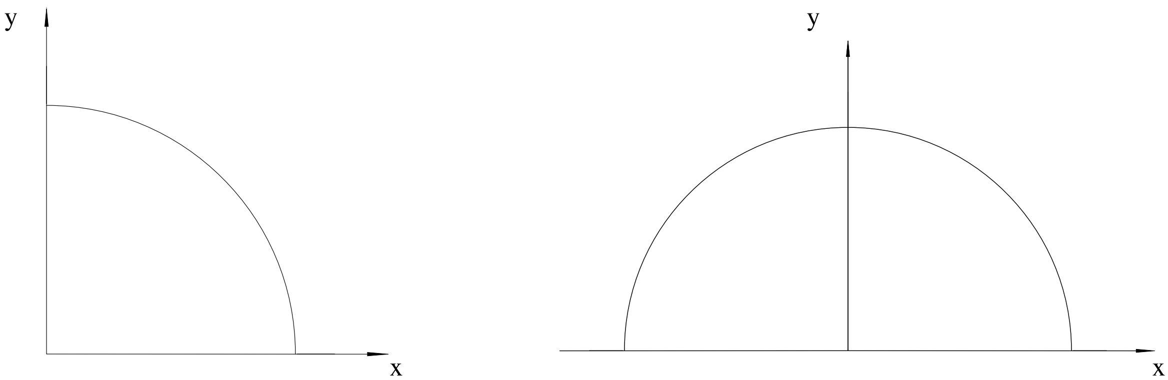
$$D_{i,j} = \int_0^1 \int_0^1 G_{k,l,n,m}(x,y)\, \mathrm{d}y\, \mathrm{d}x + \int_0^1 \int_{y_c(x)}^1 G_{k,l,n,m}(x,y)\, \mathrm{d}y\, \mathrm{d}x \tag{34}$$

$$G_{k,l,n,m} = \left[X_n\, X_m\, \frac{\partial Y_l}{\partial y}\, \frac{\partial Y_k}{\partial y} + \left(Y_l\, \frac{\partial X_n}{\partial x} + X_n\, \frac{\partial Y_l}{\partial x} \right) \left(Y_k\, \frac{\partial X_m}{\partial x} + X_m\, \frac{\partial Y_k}{\partial x} \right) \right] \tag{35}$$

The calculation of this coefficient can be partially solved analytically, so the function **Integrate** from the *Mathematica* software is required. Although not all calculation can be done analytically, it can be concluded that the implementation of this coefficient has a lower computation cost when considered to the CDA scheme, which is purely numerically obtained.

4. RESULTS AND DISCUSSION

One important remark before the presentation of the results is that, for both CDA and FDA, the solution will be presented in semi-circular regions with $\phi = 90°$ and $\phi = 180°$, as described in figure 2.



(a) $\phi = 90°$ (b) $\phi = 180°$

Figure 2: Considered geometries for convergence analysis.

The first set of results is presented for the semi-circular domain with the Coincident Domain Approach, as shown in table 1, where the analytical value in cylindrical coordinates is considered. Considering that the converged eigenvalue is the analytical solution in cylindrical coordinates, one can infer that a maximum of 6 fully converged digits were obtained with as low as 100 terms for some eigenvalues, showing satisfactory error estimates.

For the FDA scheme, the solution can be compared with different values of k_{fic}. In order to acquire the convergence behavior, values of $k_{fic} = 10$ and 100 are investigated in tables 2 and 3. As can be seen, the convergence for the FDA is clearly worse than the CDA with a maximum of 2 fully converged digits obtained in some cases. Specifically, as higher values of k_{fic} are considered, the convergence analysis gets worse with the same number of truncation orders than for smaller values of k_{fic}. However, a clear path can be delimited: smaller values of k_{fic} converge to distant values from the analytical solution, as for higher values of k_{fic} need more terms to converge, but the values seem to get closer to the analytical ones. If a triangular would to be consider, the whole implementation would be purely analytical, so more terms can be described. Although a worse convergence is obtained, the FDA has a advantage towards the CDA, which is that a irregular domain with a cavity or hole can be described only in FDA scheme.

Table 1: Convergence behavior in a semi-circular geometry with Dirichlet Boundary Conditions – CDA.

$\phi = 90°$										
k_{max}	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}
20	5.13832	7.60342	8.42181	9.98437	11.09534	11.62904	12.35295	13.66862	14.49606	14.79870
40	5.13652	7.59344	8.41873	9.95080	11.07236	11.62193	12.26449	13.61275	14.38323	14.56280
60	5.13612	7.59124	8.41806	9.94424	11.06902	11.62097	12.24140	13.60093	14.37814	14.51187
80	5.13596	7.59026	8.41780	9.94149	11.06752	11.62061	12.23631	13.59679	14.37604	14.49716
100	5.13585	7.58966	8.41762	9.93983	11.06664	11.62037	12.23280	13.59444	14.37495	14.48972
250	5.13568	7.58868	8.41734	9.93708	11.06519	11.61997	12.22716	13.59061	14.37315	14.47916
500	5.13564	7.58845	8.41728	9.93644	11.06487	11.61988	12.22580	13.58974	14.37275	14.47680
750	5.13563	7.58840	8.41726	9.93628	11.06479	11.61986	12.22546	13.58952	14.37264	14.47618
1000	5.13563	7.58838	8.41725	9.93622	11.06476	11.61986	12.22533	13.58944	14.37261	14.47594
∞	5.13562	7.58834	8.41724	9.93611	11.06471	11.61984	12.22509	13.58892	14.37254	14.47550
$\phi = 180°$										
k_{max}	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}
20	3.83329	5.14647	6.40266	7.01908	7.64424	8.44744	8.85079	9.79459	10.21495	10.24946
40	3.83238	5.13849	6.38917	7.01688	7.60372	8.42201	8.80449	9.77749	9.98520	10.17626
60	3.83203	5.13719	6.38482	7.01619	7.59711	8.41983	8.78782	9.76861	9.96227	10.17441
80	3.83192	5.13672	6.38293	7.01598	7.59445	8.41906	8.78148	9.76539	9.95280	10.17406
100	3.83187	5.13632	6.38231	7.01588	7.59229	8.41840	8.77911	9.76435	9.94730	10.17391
250	3.83174	5.13580	6.38066	7.01565	7.58935	8.41753	8.77332	9.76178	9.93900	10.17356
500	3.83172	5.13568	6.38033	7.01561	7.58869	8.41734	8.77211	9.76127	9.93711	10.17350
750	3.83171	5.13565	6.38025	7.01560	7.58852	8.41729	8.77181	9.76116	9.93663	10.17348
1000	3.83171	5.13564	6.38022	7.01559	7.58846	8.41728	8.77169	9.76111	9.93645	10.17348
∞	3.83171	5.13562	6.38016	7.01559	7.58834	8.41724	8.77148	9.76102	9.93611	10.17350

Table 2: Convergence in a semi-circular geometry with $\phi = 90°$ – FDA.

$\phi = 90°$										
$k = 10$										
k_{max}	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}
20	5.27685	7.83851	8.67109	10.29226	11.48134	12.04082	12.78651	14.19035	15.10643	15.52958
40	5.21015	7.74714	8.53850	10.14279	11.30977	11.78697	12.52116	13.87470	14.72583	14.87744
60	5.17914	7.70698	8.48266	10.08553	11.24295	11.70077	12.44817	13.78430	14.61807	14.75799
80	5.16458	7.68162	8.45777	10.04950	11.20208	11.66363	12.39344	13.72910	14.55695	14.70063
100	5.15174	7.66188	8.43584	10.02730	11.17103	11.63152	12.36431	13.69554	14.51301	14.66209
250	5.12034	7.61689	8.38361	9.96468	11.10302	11.55888	12.27967	13.60807	14.42004	14.55194
500	5.10395	7.59215	8.35679	9.93259	11.06636	11.52218	12.23905	13.56397	14.37164	14.50103
750	5.09650	7.58158	8.34466	9.91922	11.05079	11.50580	12.22161	13.54584	14.35137	14.48092
1000	5.09200	7.57489	8.33734	9.91077	11.04099	11.49590	12.21107	13.53439	14.33865	14.46838
∞	5.13562	7.58834	8.41724	9.93611	11.06471	11.61984	12.22509	13.58892	14.37254	14.47550
$k = 100$										
k_{max}	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}
20	5.58126	8.29483	9.20839	10.89378	12.25162	12.89861	13.54652	15.46478	19.50706	22.92077
40	5.44814	8.05164	8.94860	10.55835	11.77367	12.39922	13.02801	14.50640	15.38266	15.50791
60	5.40333	7.96057	8.86742	10.43190	11.62541	12.26992	12.84309	14.29839	15.13977	15.25360
80	5.35507	7.91061	8.78340	10.34984	11.54597	12.13932	12.75049	14.17143	15.02073	15.11774
100	5.33305	7.87407	8.74540	10.30549	11.48857	12.08280	12.68417	14.10413	14.93930	15.05078
250	5.26019	7.77032	8.62257	10.16860	11.33202	11.90571	12.50861	13.90984	14.72359	14.81602
500	5.22092	7.71559	8.55740	10.09871	11.25092	11.81403	12.42590	13.81256	14.61565	14.71291
750	5.20426	7.69210	8.52990	10.06859	11.21637	11.77568	12.38903	13.77096	14.57019	14.66922
1000	5.19389	7.67743	8.51282	10.05025	11.19484	11.75194	12.36615	13.74568	14.54198	14.64291
∞	5.13562	7.58834	8.41724	9.93611	11.06471	11.61984	12.22509	13.58892	14.37254	14.47550

Table 3: Convergence in a semi-circular geometry with $\phi = 180^\circ$ – FDA.

$\phi = 180^\circ$										
$k = 10$										
k_{max}	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}
20	4.00924	5.34352	6.75114	7.40540	8.06183	8.84029	9.40971	10.78878	10.83126	12.34792
40	3.92992	5.28781	6.56314	7.20502	7.88639	8.69281	9.04732	10.07393	10.38591	10.47497
60	3.90923	5.23346	6.51335	7.16179	7.79084	8.58228	8.98431	9.97501	10.21098	10.39627
80	3.89664	5.21272	6.48971	7.13668	7.75720	8.54302	8.93693	9.93103	10.14743	10.35489
100	3.88688	5.19470	6.46981	7.11770	7.72934	8.51068	8.90750	9.89606	10.12390	10.32479
250	3.85515	5.14395	6.42022	7.05725	7.65206	8.42271	8.83362	9.81425	10.01184	10.23292
500	3.83893	5.12042	6.39220	7.02730	7.61729	8.38372	8.79482	9.76945	9.96568	10.18960
750	3.83148	5.11059	6.37971	7.01364	7.60193	8.36763	8.77724	9.74989	9.94612	10.17002
1000	3.82662	5.10438	6.37175	7.00475	7.59323	8.35748	8.76648	9.73740	9.93393	10.15743
∞	3.83171	5.13562	6.38016	7.01559	7.58834	8.41724	8.77148	9.76102	9.93611	10.17350
$k = 100$										
k_{max}	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}
20	4.31085	5.84473	7.25384	8.04603	8.59306	10.13260	11.94728	14.79482	17.68540	19.92979
40	4.16520	5.58693	6.97770	7.66920	8.29973	9.22093	9.66237	10.78490	11.05393	11.29099
60	4.10267	5.53623	6.83554	7.53353	8.15258	9.11942	9.39369	10.50216	10.70373	10.99601
80	4.06463	5.47314	6.77626	7.45508	8.10143	8.99552	9.33970	10.39134	10.58283	10.85016
100	4.03529	5.41877	6.72310	7.39714	7.99636	8.89577	9.25475	10.30210	10.47652	10.75046
250	3.95842	5.31570	6.58937	7.24952	7.85385	8.71591	9.06233	10.08458	10.27173	10.51795
500	3.92011	5.26139	6.52958	7.17814	7.77259	8.62455	8.97654	9.99075	10.17218	10.41100
750	3.90323	5.23614	6.49950	7.14689	7.73726	8.58258	8.93721	9.94420	10.12641	10.36471
1000	3.89342	5.22291	6.48320	7.12880	7.71765	8.56069	8.91451	9.91902	10.10209	10.33816
∞	3.83171	5.13562	6.38016	7.01559	7.58834	8.41724	8.77148	9.76102	9.93611	10.17350

5. CONCLUSIONS

This paper proposed two schemes to acquire eigenproblem solutions in irregularly-shaped domains, using a semi-circular region as test-case. As a general representation of the problem is defined for both strategies, specific formulations were developed for the considered test-case. The CDA and the FDA were implemented based on the Dirichlet Boundary Conditions and a convergence investigation was conducted for both approaches.

As the results were presented, one important trend observed was that the CDA scheme presented itself as a better fit for the considered test-case within a semi-circular region in cartesian coordinates. However, the proposed methodology for CDA cannot be exploited for region with orifices, as the FDA methodology can. One strategy to consider to improve the solution methodology for the FDA requires that the integral coefficient could be calculated via analytical implementation, which would result in more terms to be consider and better error estimates for the FDA.

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