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APPLYING THE MONTE CARLO λ-NEUMANN MODEL TO STOCHASTIC REACTION-DIFFUSION PROBLEMS

Roberto Mauro Felix Squarcio
Claudio Roberto Ávila da Silva Junior
Federal Technological University of Paraná, PPGEM, Rua Deputado Heitor Alencar Furtado, 5000 Curitiba, PR, Brazil
roberto_squarcio@yahoo.com.br; avila@utfpr.edu.br.

Abstract. *Reliability theory had its origins in the mid-1940s and has been used to predict, for example, the expected life or average life of electronic and mechanical systems or components. It has been developed using concepts and definitions related to areas of probability, statistics, and mathematical optimization, among others. On this line of thought, the current work aims at studying and proposing new methodologies and numerical strategies for quantifying uncertainty in physical problems described by ordinary differential equations. In order to obtain uncertainty associated to responses to those equations, in general, the Monte Carlo simulation is used, which may generate high computational costs making its application prohibitive regarding some problems. Recent works propose methodologies - based on Neumann expansion - that have presented a satisfactory performance. Among those methodologies, the Monte Carlo λ-Neumann model introduces the λ-convergence parameter, which, besides reducing computational time, presents non-intrusive characteristics, that is, repetitive iterations with a restart of the programming routine. Numerical results are obtained for stochastic models of heat and mass transfer associated to reaction and diffusion terms with Dirichlet boundary conditions.*

Keywords: *Numerical Methods, Stochastic computation, Monte Carlo Method, Neumann Expansion.*

1. INTRODUCTION

The technological evolution witnessed in recent decades present relevant results regarding the reliability of structural systems that are very often complex and whose functionality depends heavily on the ability to predict their performance, in some cases, still subject to not-fully-controlled conditions, that is, seismic loads, noises or random loads.

On the other hand, structural reliability consists of the ability of a system to meet a required function when submitted to certain conditions and during a given period of time. In statistical terms, reliability is interpreted as the probability of a system to attain success with regard to a performance measure.

Stochastic mechanics seeks to quantify the variability of responses associated to the random entry data on the understanding that the process is a result of spatial variability. This variation forms a multivariate stochastic field, where one seeks to evaluate the first- and second-order statistical moments of responses. In this sense, Benjamin and Cornell (1970), Vanmarcke (1983), and Wang *et. al.* (2013) state that, to formulate the problem, first the properties of the system must be adequately modeled through a probability distribution and, at a second stage, the differential equation must solved aiming at obtaining the statistic moments associated to the variability of responses.

Ghanem and Spanos (1991) also present a methodology for evaluating random variables in which the random parameters of the system are modeled within a second-order scholastic process defined by its averages and covariance function. When applied to finite elements, the differential-equation solution is obtained by functionals that act as a linear or non-linear filter on the process.

In the variational formulation, the numerical solution of differential equations requires the inversion of a stochastic matrix, that is, a matrix that contains the elements associated to uncertainties of the random variables of the process, which is usually performed by Monte Carlo simulation, where the random variables are multiple and, in order to account for the result, at each iteration one has the sum of the random values associated to those variables. Otherwise, MCS transforms the problem of evaluating a definite integral into the statistical problem of estimating an average.

Another matrix inversion numerical model is presented by Shinozuka and Nomoto (1980), Adomian and Malakian (1980), and Shinozuka (1987), who apply Neumann expansion as an iterative process for solving linear algebraic equations assuming that the spectral radius of the iterative matrix is smaller than 1. Yamazaki *et. al.* (1988) apply

Neumann expansion to the stochastic finite element model with a null first-order moment and a unitary second-order moment. The correlation function is formed by a spectral density function based on the wave number vector. In each iteration, the covariance function is obtained from the geometric distance between the centroids of each element. This prerogative constitutes a system of equations that must ensure the orthogonalization and normalization of those vectors. It consists of the application of matrix standards obtained from the admitted tolerance in the process. The literature offers MCS models that actualize the matrix inversion using Cholesky decomposition. In this numerical model, the factorization components comply with a normal distribution.

In the Neumann series model, the matrix inversion is centered on its eigenvalues and satisfies a convergence criterion based on the admissible error. In Shinozuka and Deodatis (1986), the method is also applied when dealing with response variability as a result of spatial variability. Multi-dimensional and multivariate fields used in the digital generation of its sampling functions are presented by Shinozuka and Astill (1972).

Ávila and Beck (2015-April) propose the introduction to λ-parameter, which accelerates the Neumann series convergence optimizing the set of solutions through finite-dimensional linear operator norms such as Euclidean, infinity, Frobenius, and maximum norms. The efficiency of the proposal is observed in its non-intrusive characteristics and also in its substantial reduction of data processing time. The authors obtain numerical results for linear and non-linear stochastic systems referencing the method with regard to the single Monte Carlo simulation.

The aim of the present work is to lay the foundations of the MCS N-λ method for the heat and mass transfer problem with reaction and diffusion coefficients. Initially, the uncertainty results are presented using deterministic models of numerical solution, Finite Differences and Galerkin Finite Element, and Gauss quadrature.

The Monte Carlo simulation is applied to concentration fields (mass per volume) determining its first- and second-order moments. Subsequently, the Neumann expansion matrix inversion model and the proposal of introducing the λ-convergence parameter are evaluated, and uncertainty results associated to random variables - namely, the diffusion coefficient and the reaction coefficient - are obtained.

2. FORMULATION OF THE STOCHASTIC REACTION-DIFFUSION PROBLEM

The deterministic problem of heat and mass transfer, the strong formulation involving diffusion $\kappa(x)$ and reaction $\sigma(x)$ coefficients are extensively studied in present-day literature.

On the other hand, in the light of Lax-Milgram's lemma, the random problem ensures the existence, unicity, and continuity of the solution. In this sense, it is necessary that coefficients be strictly positive with compact support such that $\exists\,\underline{\alpha},\overline{\alpha}\in R^{+}\setminus\{0\},|\underline{\alpha},\overline{\alpha}|<\infty, P(\{\omega\in\Omega:\kappa(x,\omega)\in[\underline{\alpha},\overline{\alpha}],\forall\,x\in(0,l)\}=1)$ and $\exists\,\underline{\beta},\overline{\beta}\in R^{+}\setminus\{0\},\left|\underline{\beta},\overline{\beta}\right|<\infty, P\left(\left\{\omega\in\Omega:\sigma(x,\omega)\in\left[\underline{\beta},\overline{\beta}\right],\forall\,x\in(0,l)\right\}=1\right)$ where ω_k, is the k^{-th} single event, or actualization of the stochastic problem. Besides, the source term must have a finite variance, such that $q\in L^2\left(\Omega,q,P;L^2(0,l)\right)$.

In this sense, the relations between random variables are described using the triple $(\Omega, \mathcal{F}, \mathcal{P})$ where Ω is the space of events, $\mathcal{F}$ is a σ-algebra, and $\mathcal{P}$ is the probability measure. In this context, the strong form of the stochastic reaction-diffusion phenomenon is defined through the following Boundary Value Problem (BVP). Find $u(x,\omega)$, the defined concentration vector in $x\in(0,l)$, such that:

$$\text{(P.1)}\quad\begin{cases}-\dfrac{\mathrm{d}}{\mathrm{dx}}\left(\kappa\dfrac{\mathrm{du}}{\mathrm{dx}}\right)(\mathrm{x},\omega)+(\sigma u)(\mathrm{x},\omega)=\mathrm{q}(\mathrm{x},\omega),\ \forall\mathrm{x},\omega\in(0,\mathrm{l})\times(\Omega,\mathrm{F},\mathrm{P}),\\ \mathrm{u}(0,\omega)=\mathrm{u}(\mathrm{l},\omega)=0,\forall\omega\in(\Omega,\mathrm{F},\mathrm{P}).\end{cases}$$

The Galerkin method is used to obtain numerical solutions and stochastic differential equations based on the weak formulation corresponding to the BVP strong formulation. Such transformation allows reducing the regularities required for the numerical solution.

2.1 – Galerkin Stochastic Finite Element

After establishing $\omega\in(\Omega, \mathcal{F}, \mathcal{P})$, the model is used for solving the stochastic problem integrating directly to its expression, that is, by taking $v\in V$ as an approximation function and expanding the derivative product in the solution, one has:

$$\int_{\Omega}\int_{a}^{b}\left(\kappa\frac{du}{dx}\frac{dv}{dx}\right)(x,\omega)dxdP+\int_{\Omega}\int_{a}^{b}(\sigma uv)(x,\omega)dxdP=\int_{\Omega}\int_{a}^{b}(qv)(x,\omega)dxdP\,. \tag{1}$$

Having defined the equation sampling space, the expression of the problem in its bilinear form becomes: Determine $u(x,\omega_i) \in V$, in $a(u(.,\omega_i),v) = l(v), \forall v \in V,$ such that,

$$\text{(P.2)} \quad \begin{cases} a_{\omega_k}(u(\omega), v(\omega)) = \displaystyle\int_{x_e}^{x_{e+1}} \left[\kappa \frac{dv}{dx}\frac{du}{dx} + (\sigma uv) \right](x,\omega)dx, \\ l(v(\omega)) = -\displaystyle\int_{x_e}^{x_{e+1}} qv dx, \end{cases}$$

where operator $a_{\omega_k}(v,u)$ is symmetrical and positive-definite, x_e, the position of element *e*, and index k referring to the $k^{\text{-th}}$ actualization of the integration model.

The present model is an integral part of the stochastic process, and the Galerkin method is usually used to estimate the uncertainties associated to the concentration (mass per volume) in the diffusion-reaction problem. This approximation presents the following form:

$$u(x,\omega) = \sum_{i=1}^{\infty} u_i \phi_i(x,\omega), \tag{2}$$

where $u_i \in R, \forall i \in N$ are the coefficients to be determined and ϕ_i are the test functions - the Hermite polynomials among them.

In order that the integration of the problem makes sense, the interpolation functions must have a continuous derivative or a piecewise continuous, that is, it must belong to a class $C^1(0,l)$. In addition, due to the fact the residue is orthogonal to any base function, imposing an orthonormality and neglecting the terms relative to the boundary conditions, one has:

$$a_{\omega_k}(\phi_i(x), \phi_j(x)) = \int_\Omega \int_a^b \left(\kappa \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} \right)(x,\omega)dxdP + \int_\Omega \int_a^b (\sigma\phi_i\phi_j)(x,\omega)dxdP. \tag{3}$$

and

$$l_i^k(\phi_i(x)) = -\int_a^b (\phi_i q)(x,\omega)dx, \tag{4}$$

where $\phi_i(x), \phi_j(x)$ are the approximation functions, κ and σ assume their average values.

The abstract variational problem (P.2) is transformed into the proposal below using a system of linear equations: Find $U = \{u_i\}_{i=1}^M$ with $u_i \in R, \forall i \in \{1,\ldots,M\}$, such that:

$$\left\{ \sum_{i=1}^{M} \left[a_{\omega_k}(\phi_i(x), \phi_j(x)) \right] \right\} u_i = l_i^k(\phi_i(x)). \tag{5}$$

This problem allows visualizing the formal relation $[k_{ij}]\{u_j\} = \{q_j\}$, defined in the form of vectors and matrices, where $K \in M_M(R)$. The elements of this matrix are estimated for a fixed position of vector *x* and are expressed based on the variability of the stochastic process, as follows:

$$k(x,\omega) = \mu_k(x) + \sum_{q=1}^{N} \xi_q(\omega)\varphi_q(x), \tag{6}$$

where $\mu_k(x)$ represents the average of *k*, $\xi_q(\omega_k)$ represents its probability density function, and $\varphi_q(x)$ represents the approximation function.

Thus,

$$a_{\varpi_k}(u,v) = k_{ij}^0 u_{ij}^0(\omega_k) + \sum_{q=1}^{N} \xi_q(\omega_k) k_{ij}^q u_{ij}^q(\omega_k), \tag{7}$$

where:

$$\begin{cases} k_{ij}^0 = \int_a^b \left(\mu_k \dfrac{d\phi_i}{dx} \dfrac{d\phi_j}{dx} \right) dx, \\ k_{ij}^q = \int_a^b \left(\varphi_q \dfrac{d\phi_i}{dx} \dfrac{d\phi_j}{dx} \right) dx. \end{cases} \tag{8}$$

This way, the BVP may be rewritten as follows: Determine $u(x,\omega_i) \in V$, such that,

(P.3) $$\begin{cases} a_{\varpi_k}(u,v) = l(v), \forall v \in V, \\ \sum_{i=1}^{n} k_{ij}^0 u_i(\omega_k) + \sum_{q=1}^{N} \xi_q(\omega_k) \left[\sum_{i=1}^{n} k_{ij}^q u_i(\omega_k) \right] = q_i(v) \end{cases}$$

The inversion of the stiffness matrix demands considerable computational time and, thus, the usefulness of the Neumann series is gradually standing out. The operational cost of this inversion is minimized without compromising the accuracy of the results. The method presented in the following section is used to perform this random operation using Neumann expansion and, subsequently, the introduction of the numerical optimization solution and the convergence parameter.

2.2 – Neumann Expansion and Parameter *λ*

Yamazaki *et. al.* (1985), Shinozuka and Nomoto (1980), Adomian and Malakian (1980), Shinozuka (1987), and Ávila and Beck (2015) propose that, in order to actualize this operation, one has to rewrite the P.3 equation applying Neumann series, where matrix *K* is decomposed as follows:

$$\left[K^0 + \sum_{q=1}^{N} \xi_q(x,\omega_k) K^q \right] U(x,\omega_k) = q_i(x,\omega) \tag{9}$$

Therefore, regarding the concentration vector, one has:

$$U(x,\omega_k) = [K(x,\omega_k)]^{-1} q_i(x,\omega). \tag{10}$$

The variational formulation of the method was carried out using the Neumann series to solve the linear system below, for $U_0 : (\Omega, F, P) \to R^n$, described by: Determine $U(\xi_k) \in (\Omega, F, P)$, such that,

(P.4) $$\begin{cases} U(\xi_k) = (I + P(\xi_k))^{-1} U_0, \\ U_0(\xi_{\alpha\beta}) = K_0^{-1}(\xi_{\alpha\beta}) Q_0, \end{cases}$$

where *P* is the argument of the Neumann series, $U_0 = (K_0)^1 q$, $U_0 = [u_1^0, K, u_m^0]^T$, and the boundary conditions applied to the system.

The concepts established by Ávila and Beck (2015) consider that matrix *K*, that is, the operator for the k^{th} the coefficient sample, admits the following decomposition.

$$K(x,\xi_{\alpha\beta}) = K_0 + \Delta K(x,\xi_{\alpha\beta}), \tag{11}$$

where *ΔK* is the uncertainly associated to matrix *K,* in position *x* and for the polynomial function *ξ(ω),* given by:

$$\Delta K = \sum_{q=1}^{N} \xi_q(x, \omega_k) K^q. \tag{12}$$

The matrix entries are evaluated in the expected values of the reaction and diffusion coefficients, whereas the entries of matrix $\Delta K(x, \xi_{\alpha\beta})$ are calculated based on the random variability, around the expected values of those coefficients. Based on the argument of one Neumann series, $P(x, \xi_{\alpha\beta}) = K_0^{-1} \Delta K(x, \xi_{\alpha\beta})$, the matrix can be written as:

$$K(\xi_{\alpha\beta}) = K_0 (I - P(\xi_{\alpha\beta})). \tag{13}$$

where *I* is the identity matrix. Thus, the concentration vector is estimated based on:

$$U(\xi_{\alpha\beta}) = K^{-1}(\xi_{\alpha\beta}) q = (I - P(\xi_{\alpha\beta}))^{-1} U_0. \tag{14}$$

By the definition and properties related to the Neumann series, that is, $\|P\| < 1$ e $(I - P(\omega_k))^{-1} = \sum_{s=0}^{\infty} P^s = I + P + P^2 + ...$ one observes that the order-*M* partial sum is expressed by:

$$(I - P(\varpi_k))_M^{-1} = \sum_{s=0}^{M} P^S. \tag{15}$$

Equation (14) allows introducing the admitted residue matrix, ε, such that, $(I - P)(I - P)_k^{-1} = I - \varepsilon$. Ávila and Beck (2015) demonstrate that the admitted tolerance can be estimated by linear approximation defined in Matrix Norms dealing with the problem using the definition of parameter λ, with components, λ_1 and λ_2, such that:

$$(I - P)(I - P)_{(k)}^{-1} = \lambda_1 I + \lambda_2 P. \tag{16}$$

For a first-order approximation, the following optimization problem can be established: Find $(\lambda_1^*, \lambda_2^*) \in R^2$, such that,

$$\text{(P.5)} \begin{cases} (\lambda_1^*, \lambda_2^*) = \underset{(\lambda_1, \lambda_2) \in R^2}{\arg\min} \left\{ \frac{1}{2} \| \lambda_1 (I - P(\xi_{kn})) + \lambda_2 P(\xi_{kn})(I - P(\xi_{kn})) \|^2 \right\}. \\ (I - P)(I - P)_k^{-1} = I - \varepsilon. \end{cases}$$

The objective function is non-negative and convex, and the global optimum $(\lambda_1^*, \lambda_2^*)$ is obtained by stationarity conditions, as described in Nocedal (1999), that is, $\nabla_\lambda f(\lambda_1^*, \lambda_2^*) = 0$:

$$\begin{cases} \lambda_1^* = \dfrac{\phi - \varphi \lambda_2^*}{\alpha}, \\ \lambda_2^* = \dfrac{\alpha \vartheta - \varphi \phi}{\alpha \gamma - \varphi^2}, \end{cases} \tag{17}$$

where:

$$\begin{aligned} \alpha(\xi_{kn}) &= ((I - P(\xi_{kn}))U_0)^T (I - P(\xi_{kn}))U_0; \\ \varphi(\xi_{kn}) &= ((I - P(\xi_{kn}))U_0)^T P(\xi_{kn})(I - P(\xi_{kn}))U_0; \\ \phi(\xi_{kn}) &= U_0^T (I - P(\xi_{kn}))U_0; \\ \gamma(\xi_{kn}) &= (P(\xi_{kn})(I - P(\xi_{kn}))U_0)^T P(\xi_{kn})(I - P(\xi_{kn}))U_0; \\ \vartheta(\xi_{kn}) &= U_0^T P(\xi_{kn})(I - P(\xi_{kn}))U_0; \end{aligned} \tag{18}$$

Thus, the proposal of the MCS N-λ method establishes that the k^{th} actualization of the linear system of Eq. (13) through Neumann expansion, with $n = 1$, is given by:

$$U_{(1)}\left(\lambda^*, \xi_{\alpha\beta}\right) = \left(\lambda_1^* I + \lambda_2^* P\left(\xi_{\alpha\beta}\right)\right) U_0 \tag{19}$$

These results are applied considering the problem of heat and mass transfer with associated uncertainties to the reaction and diffusion coefficients.

3. NUMERICAL RESULTS

In the present section, the accuracy and performance of the MC-N λ method are evaluated regarding a reaction-diffusion problem in a continuous medium measuring *l = 1 m* long, with known concentration values at both ends. The reaction and diffusion coefficients are assumed as random variables. Both coefficients are modeled as a parameterized random process. The problem is formulated considering Dirichlet conditions, $u(0,\omega) = u(l,\omega) = 0$, where:

$$k(x) = k_0 + k_1 \cos\left(\frac{\pi . x}{l}\right) \text{ and} \tag{20}$$

$$\lambda(x) = \lambda_0 + \lambda_1 \cos\left(\frac{\pi . x}{l}\right), \tag{21}$$

attributing values $k_0 = 250000$, $k_1 = k_0/2$, $\lambda_0 = 16800$, $\lambda_1 = \lambda_0/2$ $l = 1$. The source term is assumed in the distributed form, $f(x) = \sin(x)$. Uncertainties associated to the random vectors are obtained through the estimators of their statistical moments, that is,

$$\begin{aligned} \hat{\mu}_{u(x)} &= \frac{1}{N}\sum_{i=1}^{N} u(x,\omega_i); \\ \hat{\sigma}^2_{u(x)} &= \left(\frac{1}{N-1}\right)\sum_{i=1}^{N}\left[u(x,\omega_i) - \hat{\mu}_{u(x)}\right]^2. \end{aligned} \tag{22}$$

where $\hat{\mu}_{u(x)}$ is the average of a concentration vector sample *u(x)* and $\hat{\sigma}^2_{u(x)}$ is the variance of *u(x)*.

Comparative results are obtained based on the Finite Difference Method and the Finite Element Method with variational integration actualized by Galerkin and Gaussian Quadrature. Figure 1 presents the deterministic results for concentration *u(x)* along length *l*. For FDM, a 1000-iteration discretization is used. For FEM, the results presented are obtained with 10 elements. The solution per quadrature used the two-point Gauss-Legendre formula.

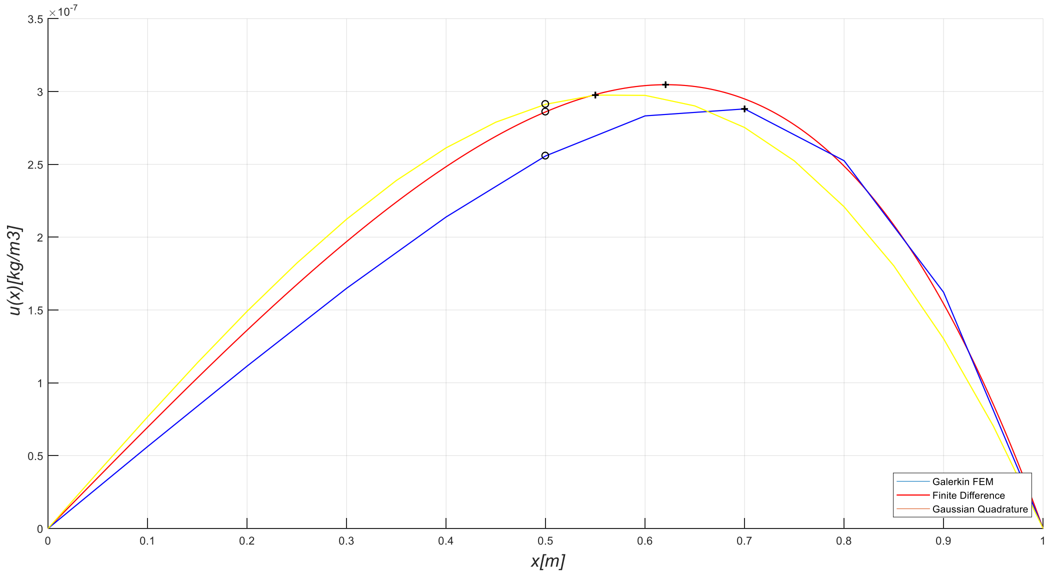

Figure 1: Average value of concentration *u(x)* obtained for FDM and FEM methods.

The uncertainty of the coefficients is presented using the parameterized stochastic process formed as a linear combination of continuous and statistically independent functions with evenly distributed random variables: $\xi_i : (\Omega, F, P) \rightarrow [-1,1]$, *i = 1,2,* with expected value and variance given by:

$$\begin{cases} \hat{\mu}_{\xi_i} = 0; \\ \hat{\sigma}^2_{\xi_i} = 1. \end{cases}$$

Approximate solutions are obtained for statistics when *x = 0.5 m*, that is, $u(0.5, \omega)$. Figure 2 illustrates the histogram and adjustment of the concentration, through FDM, Galerkin, and Gaussian quadrature, for that position.

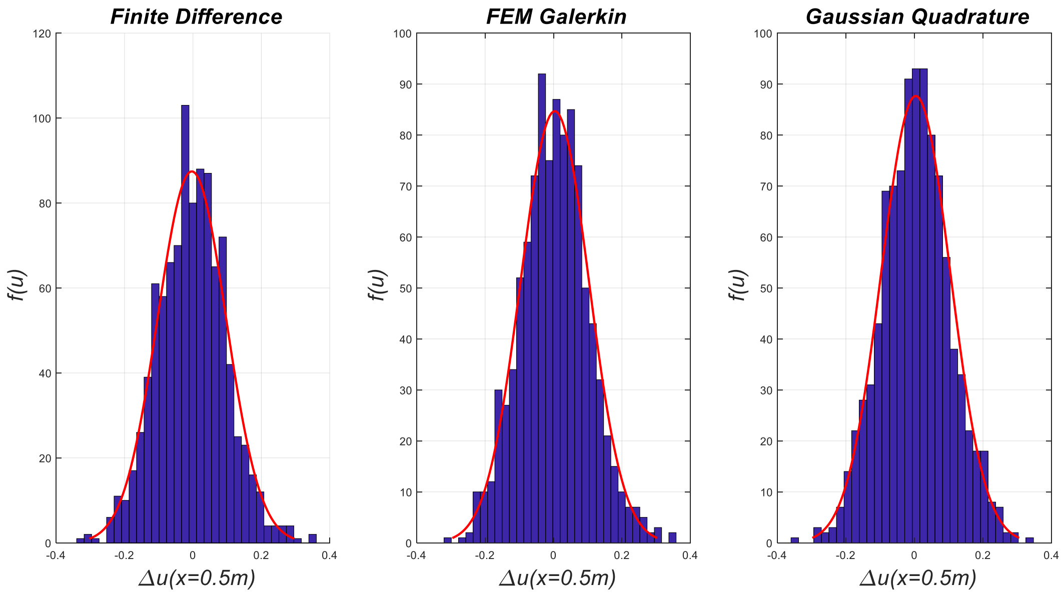

Figure 2. Histogram of the concentration – Monte Carlo Simulation

One observes that not only the first- and second-order statistical moments are obtained, but also the complete probability distribution of the response.

Table 1 presents parameterized results associated to the concentration (mass per unit of volume) through the MCS in the three models that were applied.

Table 1. Uncertainty results, by MCS, of displacement in *x = 0.5 m*.

	MC-FD	MC-Galerkin	MC-Gauss
u maximum (10^{-7}m)	0.33306	0.32124	0.26636
u minimum (10^{-7}m)	-0.27826	-0.33876	-0.31217
variation (10^{-7}m)	0.61132	0.66000	0.57853

Table 2 presents parameterized results of the concentration using MCS N-λ in the deterministic models.

Table 2. Uncertainty results, by MCS N-*λ*, of the displacement in *x = 0.5 m*.

	N-FD	N-Galerkin	N-Gauss
u maximum (10^{-7}m)	0.32954	0.29050	0.28357
u minimum (10^{-7}m)	-0.25328	-0.33010	-0.26477
u average (10^{-7}m)	0.28552	0.25787	0.29063

Two values of variation coefficient are considered: $\delta = 1/10$ and $\delta = 3/10$. For both coefficients, the estimate of the expected value and variance are obtained for a number of thirty thousand actualizations, but one observes that, based on five thousand actualizations, there is a strong convergence of estimates.

Considering the above-mentioned conditions of each deterministic solution model, using a 7-generation processor, Corel i7 with a 16GB memory and a 1T hard disk, using software MatLab®, R2017 – the student version -, the processing times (in seconds) are presented in Table 3.

Table 3. Computational processing time (in seconds)

	FD	Galerkin	Gauss
MCS Direct	0.156	145.211	0.118
MCS Neumann	0.112	25.305	0.055
MCS N-λ	0.049	17.448	0.023

One observes that the time reduction proposed by the MCS N-λ model is due to a lesser computational effort regarding the non-intrusiveness of the iterative process. Neumann expansion does not make it necessary to reinitialize the iterative construction and inversion process of the stochastic matrix.

4. CONCLUSION

In the present work, the Finite Differences method, the Galerkin method, and the Gaussian Quadrature Finite Element method are used to construct the numerical solutions of the reaction-diffusion equation with uncertainties associated to the reaction and diffusion coefficients. The stochastic version of Lax-Milgram lemma is used to ensure the existence and unicity conditions of the solution to this problem. Solutions that do not comply with such conditions become weak and can return invalid results to many problems, for example, when large variation coefficients are used. In practical terms, the results presented show that, for solving stochastic numerical problems, the way of representing random problem parameters has a considerable importance.

Effective results in terms of computational time and accuracy are obtained with the first-order approximation of Neumann expansion when compared to the direct Monte Carlo simulation method. The convergence of the series is guaranteed for each sample with an argument smaller than 1, and deterministic finite-element programs are easily adapted for that purpose. For future works, one can suggest that the method be implemented for high-order solutions with a consequent increase of the vector λ dimension.

Although the deterministic estimated results of the concentration among the models showed to be very close, the statistical moments of the uncertainties associated to them presented a considerable difference. In the Galerkin method, the Hermite polynomials were used as form functions, and - for future works - it may be suggested that they be used as Airy stress functions.

In order to improve the efficiency of the present proposal, the equivalence between the finite-dimension operator norms are considered. Solutions using the MCS N-λ are calculated for five equivalent norms using one/a term of Neumann expansion. It is worth repeating here the observation that a good compromise between accuracy and efficiency is obtained through Frobenius norm.

Deviations of the Monte Carlo-Neumann and MCS N-λ are calculated in order to compare them with the reference calculated by direct MCS. The accuracy of the boundaries (maxima and minima) obtained by using the Monte Carlo proposal is attained using only one term of Neumann expansion.

The proposal of the present work shows a viable alternative for solving the problem of uncertainty propagations.

5. AKNOWLEDGEMENTS

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6. REFERENCES

Adomian, G and Malakian, K., 1980. “Inversion of stochastic partial differential operators: the linear case”, *J. Math. Anal. Appl.*, vol. 77, pp. 309-327.

Ávila S. Jr., C. R. and Beck, A. T., 2013. “Efficient Bounds for the Monte Carlo – Neumann Solution of Stochastic Thermo-Elasticity Problems”, *Journal of Heat Transfer*, v. 135, 071201-1.

Ávila S. Jr., C. R., Beck, A. T., Franco, A.T., Suarez O.A., 2015a. “Galerkin Solution of Stochastic Reaction-Diffusion Problems”, *International Journal of Solids and Structures*, v. 58, pp. 136-145.

Ávila S. Jr., C. R. and Beck, A. T., 2015b. “A Fast Convergence Parameter for Monte Carlo – Neumann Solution of Linear Stochastic Systems”, ASCE-ASME., *Journal Risk and Uncert. In Engrg. Sys.*, Part B, vol. 1, pp. 1-9.

Benjamin, J., Cornell, C., 1970. “Probability, Statistic and Decision for Civil Engineers”, McGraw-Hill, New York.

Ghanem, R.G. and Spanos, P.D., 1991. “Stochastic Finite Elements: A Spectral Approach”, Springer, New York.

Nocedal, J., Wright, S. J., 1999. “Numerical Optimization”. Springer-Verlag, New York, NY, EUA.

Shinozuka, M., 1987. “Structural response variability”, *Journal of Engineering Mechanics*, ASCE, Vol. 113, No EM6, pp. 825-842.

Shinozuka, M., Astill J., 1972. “Random eigenvalue problems in structural mechanics”, *AIAA Journal*, Vol. 10, No 4, pp. 456-462.

Shinozuka, M. and Deodatis, G., 1986. “Response Variability of Stochastic Finite Element Systems”, *Technical Report, Dept. of Civil Engineering*, Columbia University, New York.

Vanmarcke, E.H., 1983. “Randon Fields: Analysis and Synthesis”, MIT Press; Cambridge, MA.

Wang, X., Cen, S., Li, C., 2013. “Generalized Neumann Expansion and Its Application in Stochastic Finite Element Methods”, *Mathematical Problems in Engineering*.

Yamazaki, F., Shinozuka, M., Dasgupta, G., 1988. “Neumann Expansion for Stochastic Finite Element Analysis”, *Journal of Engineering Mechanics*, 114(8).

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